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S-Matrix Theory of Strong Interactions

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S-Matrix Theory of Strong Interactions

A Lecture Note and Reprint Volume

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University of California Physics Department
and Lawrence Radiation Laboratory

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S-MATRIX THEORY OF STRONG INTERACTIONS
A Lecture Note and Reprint Volume

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EDITOR'S FOREWORD

The problem of communicating in a coherent fashion the recent developments in the most exciting and active fields of physics seems particularly pressing today. The enormous growth in the number of physicists has tended to make the familiar channels of communication considerably less effective. It has become increasingly difficult for experts in a given field to keep up with the current literature; the novice can only be confused. What is needed is both a consistent account of a field and the presentation of a definite "point of view" concerning it. Formal monographs cannot meet such a need in a rapidly developing field, and, perhaps more important, the review article seems to have fallen into disfavor. Indeed, it would seem that the people most actively engaged in developing a given field are the people least likely to write at length about it.

"Frontiers in Physics" has been conceived in an effort to improve the situation in several ways. First, to take advantage of the fact that the leading physicists today frequently give a series of lectures, a graduate seminar, or a graduate course in their special fields of interest. Such lectures serve to summarize the present status of a rapidly developing field and may well constitute the only coherent account available at the time. Often, notes on lectures exist (prepared by the lecturer himself, by graduate students, or by postdoctoral fellows) and have been distributed in mimeographed form on a limited basis. One of the principal purposes of the "Frontiers in Physics" series is to make such notes available to a wider audience of physicists.

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It should be emphasized that lecture notes are necessarily rough and informal, both in style and content, and those in the series will prove no exception. This is as it should be. The point of the series is to offer new, rapid, more informal, and, it is hoped, more effective ways for physicists to teach one another. The point is lost if only elegant notes qualify.

A second way to improve communication in very active fields of physics is by the publication of collections of reprints of recent articles. Such collections are themselves useful to people working in the field. The value of the reprints would, however, seem much enhanced if the collection would be accompanied by an introduction of moderate length, which would serve to tie the collection together and, necessarily, constitute a brief survey of the present status of the field. Again, it is appropriate that such an introduction be informal, in keeping with the active character of the field.

A third possibility for the series might be called an informal monograph, to connote the fact that it represents an intermediate step between lecture notes and formal monographs. It would offer the author an opportunity to present his views of a field that has developed to the point at which a summation might prove extraordinarily fruitful, but for which a formal monograph might not be feasible or desirable.

Fourth, there are the contemporary classics—papers or lectures which constitute a particularly valuable approach to the teaching and learning of physics today. Here one thinks of fields that lie at the heart of much of present-day research, but whose essentials are by now well understood, such as quantum electrodynamics or magnetic resonance. In such fields some of the best pedagogical material is not readily available, either because it consists of papers long out of print or lectures that have never been published.

"Frontiers in Physics" is designed to be flexible in editorial format. Authors are encouraged to use as many of the foregoing approaches as seem desirable for the project at hand. The publishing format for the series is in keeping with its intentions. Photo-offset printing is used throughout, and the books are paperbound, in order to speed publication and reduce costs. It is hoped that the books will thereby be within the financial reach of graduate students in this country and abroad.

Finally, because the series represents something of an experiment on the part of the editor and the publisher, suggestions from interested readers as to format, contributors, and contributions will be most welcome.

DAVID PINES

Urbana, Illinois
August 1961

PREFACE

The material presented here originated in lectures given at summer schools in Les Houches and Edinburgh in 1960, but these notes represent an extensive revision of and addition to my contribution to the published proceedings of the seminars. The most important additions center around the double spectral function, whose role in the S matrix has been substantially clarified during the past year. This clarification has sharpened the distinction between elementary and composite particles, a distinction that is now becoming the central problem of strong-interaction physics.

Readers should be aware that S-matrix theory is still incomplete and should not expect to find a well-defined theoretical structure in these notes. Progress is currently rapid, and a complete theory may well develop within a few years' time. (A paper by H. Stapp, currently in press, makes a major step in this direction.) A monograph will then be in order, one that begins with a clear set of S-matrix postulates and makes no reference to the concept of field. The present notes, however, occasionally refer to results from field theory in order to motivate assumptions about the S matrix, even though no use is made of the field concept itself. In particular we shall find Feynman diagrams highly useful. Some students justifiably will be disturbed by the logic of such an approach, and to them I can only appeal for patience. If one wishes at the present time to work in the theory of strong interactions, it appears necessary to tolerate some obscurity in the logical foundation.

These notes are intended for students who wish to participate actively in particle-physics research; a knowledge of the basic principles of quantum mechanics and special relativity is taken for granted, as is acquaintance with Feynman diagrams. It is, however, unnecessary to be conversant with the subtleties of field theory, and a certain innocence in this respect is per-

haps even desirable. Experts in field theory seem to find current trends in S-matrix research more baffling than do nonexperts.

I regret that so little attention is given here to the crucial matter of comparison with experiment. After all, the importance of S-matrix theory stems from its success wherever experimental testing has been possible. However, most tests have been made with processes such as nucleon-nucleon scattering or photo-pion production, which are too complex to be discussed in this volume. To alleviate the deficiency, a number of references are given to the original literature on experimental verification, as well as to review articles.

With regard to the articles reprinted here, an attempt was made to select papers that present an up-to-date point of view and contain fundamental material not covered in detail in the notes. Some of these papers have historical significance in the development of S-matrix theory, but that was not the criterion for selection. Many of the most influential papers historically are not included.

GEOFFREY CHEW

Berkeley, California
August 1961

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L. D. Landau: On Analytic Properties of Vertex Parts in Quantum Field Theory • Nuclear Physics, **13**, 181-192 (1959) 101

R. E. Cutkosky: Singularities and Discontinuities of Feynman Amplitudes • J. Math. Physics, **1**, 429-433 (1960) 113

S. Mandelstam: Determination of the Pion-Nucleon Scattering Amplitude from Dispersion Relations and Unitarity: General Theory • Phys. Rev., **112**, 1344-1360 (1958) 118

T. W. B. Kibble: Kinematics of General Scattering Processes and the Mandelstam Representation • Phys. Rev., **117**, 1159-1162 (1960) 135

R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Trieman: Mandelstam Representation for Potential Scattering • Ann. Physics, **10**, 62-93 (1960) 139

T. Regge: Bound States, Shadow States and the Mandelstam Representation • Nuovo cimento, **18**, 947-956 (1960) 171

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1 HISTORICAL SURVEY AND GENERAL OUTLOOK

A sustained series of field-theoretical developments over the past decade has brought the concept of an S-matrix theory of strong interactions close to realization. Although he has long since renounced the S-matrix approach, the original work was done by Heisenberg in 1943.[1] Heisenberg lost interest, probably because in the forties he lacked the full analytic continuation that is required to give the S matrix dynamical content. He also, as is well known, subsequently abandoned conventional many-field theory and has thrown all his efforts behind the idea of a single underlying field.[2] It should be realized that the S-matrix theory of strong interactions, at least as I shall describe it in these lectures, has the same goal as the single-field approach: that is, given certain symmetries, to predict all the observed particles, together with masses and mutual interactions, in terms of a single constant with the dimensions of length. There should be no arbitrary dimensionless constants. It is conceivable, then, that the two approaches are not contradictory but complementary. Heisenberg likes to say that the one works from the inside out, the other from the outside in. He believes that simplicity lies only at the "center" while on the "periphery" there is confusion. I tend not to believe in the existence of a "center"; furthermore, on the "periphery," even though the situation is complicated, the rules are subject to a more or less direct experimental test.

So that there can be no misunderstanding of the point of view of these lectures, let me say at once that I believe the conventional association of fields with strongly interacting particles to be empty. I do not have firm convictions about leptons or photons, but it seems to me that no aspect of strong interactions has been clarified by the field concept. Whatever success theory has achieved in this area is based on the unitarity of the analytically continued S-matrix plus symmetry principles. I do not wish to assert

(as does Landau[3]) that conventional field theory is necessarily wrong, but only that it is sterile with respect to strong interactions and that, like an old soldier, it is destined not to die but just to fade away.

Having made this point so strongly, I hasten to express an unqualified appreciation of the historical role played by field theory up to the present. The field apparatus has been enormously useful in the discovery of symmetry principles, particularly with respect to charge conjugation. A second area where field theory has played a crucial historical part is in the analytic continuation of the S matrix; the notion of micro-causality and of Feynman diagrams has been invaluable in this connection. However, it is my impression now that finally we have within our grasp all the properties of the S matrix that can be deduced from field theory and that future development of an understanding of strong interactions will be expedited if we eliminate from our thinking such field-theoretical notions as Lagrangians, "bare" masses, "bare" coupling constants, and even the notion of "elementary particles." I believe, in other words, that in the future we should work entirely within the framework of the analytically continued S matrix.

The essential information about the S matrix that has been given by field-theoretical studies is the location and strength of unphysical singularities. The rule has been stated in the most complete form by Landau[4] and Cutkosky,[5] building on observations made by many others, starting in the mid-fifties. In 1955 Chew and Low[6] showed for the static model of the pion-nucleon interaction that one was dealing with an analytic function of the energy, that the "forces" could be associated with singularities of the scattering amplitude in unphysical regions, and that a knowledge of the location and strength of these singularities was probably sufficient to determine the S matrix. Furthermore, these authors showed how to calculate certain particularly important unphysical singularities and thereby successfully explained a number of experimentally observed features of the low-energy pion-nucleon system.

Of course the static model is not Lorentz-invariant, and it fails to include many interactions that must be important. That Lorentz invariance was not a difficulty was suggested by the form of relativistic fixed-momentum-transfer dispersion relations, proposed in 1955 by Goldberger[7] and by Karplus and Ruderman,[8] to which the static Low equations were shown to bear a striking resemblance.[9] These "one-dimensional" relations, however, do not describe all the unphysical singularities and are insufficient to determine the S matrix, even for elastic scattering. The inclusion of all the forces requires a knowledge of singularities in momentum transfer as well as energy; this information was provided by the double-dispersion relations, proposed in 1958 by Mandelstam.[10] A generalization of Mandelstam's ideas to elements of the S matrix involving more than two particles is required before the theory can be regarded as complete. Such a generalization has been formulated by Landau[4] and Cutkosky,[5] and their work indicates that only diligence and ingenuity is needed to handle the larger number of degrees of freedom. From a practical standpoint the one- and two-particle S-matrix

elements continue at present to saturate the theorists' capacity for calculation, but three-particle states may soon become a center of attention.

The Landau-Cutkosky recipe is couched in the language of Feynman diagrams and therefore seems to rest heavily on field theory. In the case of elastic scattering, however, Mandelstam originally discovered an important part of the recipe not through diagrams but by asking the question: Is there a way, consistent with unitarity, to continue analytically the S matrix in both angle and energy variables? He found a prescription for doing this, and no one has succeeded in finding an alternative. On the basis of this experience it is plausible that the complete Landau-Cutkosky rules may be a unique consequence of the following postulate: *The S matrix is a Lorentz-invariant analytic function of all momentum variables with only those singularities required by unitarity.* The requirement of simultaneous unitarity in all the different channels of the S matrix obtained by switching incoming and outgoing particles is an enormously restrictive one. At first glance, in fact, it sometimes seems impossible, and the only machinery we have that can contemplate such a problem with any generality is based on diagrams motivated by field theory. It appears to me nevertheless likely that the essence of the diagrammatic approach will eventually be divorced from field theory and be shown to rest only on the twin principles of analyticity and unitarity.

It is not claimed, of course, that the above postulate has been stated with precision, and one does not yet see how it produces the complete Landau-Cutkosky rules. However, the simpler aspects of these rules that have been studied in some detail appear to contain nothing superfluous with respect to analyticity and unitarity. In particular, the following aspect of the S matrix has impressed everyone who thinks in these terms: Given certain singularities and the requirement of unitarity in physical regions, the existence of other singularities is implied. Mandelstam's original work[10] was based on this circumstance, as is the possibility of predicting the existence of resonances and bound states. A more concrete way of stating the fundamental postulate, then, is to say that once one is given certain simple singularities (e.g., some of the poles), the location and strength of *all* other singularities are determined by the constraint of unitarity in physical regions. The solution of such a problem I presume to be consistent with the prescription given by Landau and Cutkosky in terms of diagrams.

Even if we assume the correctness of such a postulate, a philosophical objection may be raised against the S-matrix approach, that is, that the principle of analyticity has no physical basis, whereas in field theory it appears related to the notion of microscopic causality. My personal inclination here is to resurrect the ancient principle of "lack of sufficient reason." I assert that it is natural for an S-matrix element to vary smoothly as energies and angles are changed, and that a natural mathematical definition of physical smoothness lies in the concept of analyticity. The fundamental principle therefore might be one of maximum smoothness: The S matrix has no singularities except where absolutely necessary to satisfy unitarity. There is no "reason" for it to have any others.

Whether or not the Landau-Cutkosky rules can be derived from a principle of maximum simplicity, it seems probable that these rules must be obeyed in analytically continuing the S matrix. In these lectures we shall consider the dynamical problem on the basis of such rules. First of all, what *is* the problem? I like to state the objective in terms of the notion introduced earlier—that a knowledge of *some* singularities determines the location and strength of others. The general goal then is, given the strong-interaction symmetry principles, to make a maximum number of predictions about physical singularities in terms of a minimum amount of information about unphysical singularities. If one believes in conventional field theory, then one believes the necessary and sufficient input information to be a knowledge of the positions and residues of poles associated with "elementary particles" plus certain normalization parameters such as the pion-pion coupling constant. The poles may be on the real axis of the physical sheet (stable elementary particles) or on the unphysical sheet (unstable elementary particles). A plausibility argument that such information would indeed determine all other singularities will be made in these lectures; evidently, however, we must distinguish between elementary particles and bound states or dynamical resonances.

The only clean definition of elementary particles completely begs the question here. One must first suppose that an a priori specification of a certain minimum number of particle masses and coupling constants is in fact necessary and sufficient to determine the S matrix. All particles within this group are defined to be elementary, all others (whose masses and interactions are predictable) are either bound states, if stable, or dynamical resonances, if unstable. Such a definition is therefore meaningful only after the dynamical problem has been solved. That is to say, one does not know whether a particular particle is to be included in the select group until one has constructed the S matrix with and without an a priori specification of the corresponding poles and has compared the two results by experiment. By a reasonably simple calculation one can sometimes decide, as we shall see in these lectures, that not all of a group of neighboring poles correspond to elementary particles. It is difficult, however, to imagine a calculation sufficiently complete to approach a definite answer to the question: *Which* of the strongly interacting particles are elementary? Partly because of this circumstance, but even more because of general philosophical convictions, I am convinced that there can be only one sensible answer, and that is that *none* of them is elementary. This point of view is, of course, the basis for Heisenberg's single-field approach[2] and I am sure it is shared by many others. In particular there is a remark often made privately by Feynman that tends to convert the negative statement into a positive one. Paraphrasing Feynman: *The correct theory should be such that it does not allow one to say which particles are elementary.* Such a concept is manifestly at odds with the spirit of conventional field theory, but it forms a smooth alliance with the S-matrix approach.

For the analytically continued S matrix, the Feynman principle is simply the statement that all singularities have a common and equivalent basis. We shall see that the Landau-Cutkosky rules are in complete harmony with such a principle. They tell us that singularities occur only in connection with possible physical states, and have strengths that are determined by S-matrix elements or analytic continuations thereof. Even though these rules completely determine the dynamics, they contain not the slightest hint of a criterion for distinguishing elementary particles. We may be reminded again of the principle of "lack of sufficient reason." If one can calculate the S matrix without distinguishing elementary particles, why introduce such a notion?

Of course, without the elementary-particle concept to focus attention on particular poles of the S matrix, the question immediately arises: Where does one begin the dynamical calculation? I believe the answer to be that it doesn't matter; one may begin anywhere, taking an arbitrary singularity as a starting point and attempting to reach as much of the S matrix from this point as computational ability allows. A second question is: What determines the strength of the "starting" singularity if this strength is not controlled by a fundamental coupling constant? Here we may appeal once more to the notion of "lack of sufficient reason." The singularity strength is bounded by unitarity, so Frautschi and I have found it natural to postulate that strong interactions are characterized by "saturation" of the unitarity condition[11]; that is, they have the maximum strength consistent with unitarity and analyticity. To us there seems no *reason* for any other strength to occur, and the observed behavior of high-energy cross sections gives strong encouragement to this notion of saturation.

With such a postulate, an even clearer break is made with conventional elementary-particle field theory—where the idea of arbitrary coupling constants is usually regarded as basic. Frautschi and I believe that no arbitrary dimensionless constants occur in the strong-interaction S matrix. We are not quite so firm in our opinion about the number of dimensional constants, but it is plausible that there should be only one—to establish the scale of masses. We have absolutely no ideas as to the origin of the strong-interaction symmetries, but we expect that promising developments here can be incorporated directly into the S matrix without reference to the field concept. To summarize our conjecture, then, we believe that *all* strong-interaction physics should emerge from an analytically continued S matrix that possesses the already-recognized symmetries and "saturates" the unitarity condition. Such an S matrix is expected to depend on a single dimensional constant that may be chosen to be the mass of any one particle.

For the bulk of the material to be presented in these lectures the notion of "saturation" is not essential, and the possibility that certain particles may, after all, be more fundamental than others will be kept open. Field theory, however, will not be used. We accept the Mandelstam-Landau-Cutkosky principles as a starting point and shall investigate the consequences

for simple one- and two-body problems. Of course the dynamics will not be complete until there is a detailed calculational procedure for general S-matrix elements, involving more than two particles in both initial and final states. Such elements enter even in discussing two-body reactions because, through unitarity, they determine the "strength" of certain singularities in the two-body amplitude. How, then, can we expect to deduce any meaningful consequences from an incomplete theory? The answer rests on two general features of the Mandelstam-Landau-Cutkosky principles.

1. The location of singularities is determined by the total "masses" of actual physical systems; the higher the mass the farther from the origin is the associated singularity. Now, among the strongly interacting particles there are none of zero mass†; thus, the total "mass" of strongly interacting physical systems systematically tends to increase with the number of particles, and the singularities near the origin tend to be determined by one- and two-particle configurations. If there are aspects of the physical problem that are controlled mainly by "nearby" singularities, then one can make a meaningful comparison of theory with experiment without a complete understanding of "far-away" singularities in which multiparticle configurations play a role.

2. The "strength" of singularities is related to products of S-matrix elements and restricted by unitarity, so that in a limited region of the complex plane the behavior of an S-matrix element tends to be controlled by the closest singularities. More precisely, an analytic function is determined through the Cauchy relations by a kind of Coulomb's law for a potential due to point charges (poles) and line charges (branch cuts). The point charge is the residue of the pole, and the line-charge "density" is the discontinuity across the cut; both are proportional to products of S-matrix elements (or analytic continuations thereof) and limited in magnitude. It is assured, therefore, that the "Coulomb's law" reciprocal dependence on distance, which favors nearby singularities, will not be overwhelmed by an increasing strength of singularity with distance. From a practical standpoint, this feature of the S-matrix approach is of tremendous importance to a theory of strong interactions, permitting an orderly and systematic series of approximations whose validity is subject to a realistic appraisal without any assumption as to the magnitudes of coupling constants.

We shall see in what follows that the range of a force, from the conventional point of view, corresponds to the reciprocal distance from the origin in the complex (momentum) plane of the associated singularity. Thus the

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†Note that problems involving large numbers of low-frequency virtual photons, such as Coulomb bound states or low-velocity Coulomb scattering, cannot be handled by the approach described in these lectures. Because of the zero mass of the photon, there is no separation of single-photon and multiphoton singularities. For high particle velocities, of course, the small magnitude of the fine-structure constant often makes it possible to neglect multiple photon contributions.

"nearby" singularities, associated with one- and two-particle configurations, are the "long-range forces." The forces we cannot calculate reliably (but can only put limits on) are those of short range. This way of assessing the situation suggests the two kinds of predictions we can expect to make with the incomplete theory:

1. Scattering in states of large orbital angular momentum should be more or less completely predictable at any energy, since the centrifugal "barrier" shields these states from the unknown short-range forces. In other words, high-angular-momentum collisions are controlled by nearby singularities that our theory is able to handle.

2. In states of low angular momentum, experience with potential scattering suggests that the short-range interaction, even though complicated, and exerting a significant influence, can be represented by a small number of parameters so far as low-energy experiments are concerned. Boundary-condition treatments of the hard core in the nuclear force are based on this circumstance, as are effective-range formulas in general. The Coulomb-potential analogy to our S-matrix problem in the complex plane suggests a general explanation. Any collection of source charges (singularities), if sufficiently distant, can be replaced by a single point charge (pole) at infinity, as far as the potential (scattering amplitude) in a local region is concerned. If one wishes to represent the first derivative of the potential, which is nonzero because of the finite distance of the actual charges, an equivalent point charge at a finite distance can be found. For higher derivatives, more poles or perhaps multipoles may be added, but it is clear that far-away singularities generally produce only smooth variations and can be represented by a small number of parameters. The nearby singularities, in contrast, may be expected to produce strong and characteristic variations in the amplitude that can be identified in experimental results. These strong variations are predictable in the incomplete theory.

The inverse relation between range of interaction and distance in momentum space is of course traceable to the uncertainty principle. The unphysical singularities of an elastic-scattering amplitude correspond to the systems that can be "transferred" between the particles undergoing scattering. Only by such transfers can a force be transmitted, and it is well known that, according to the uncertainty principle, the range of the force is $\sim E^{-1}$, if E is the total energy necessary to create the transferred system. The incomplete theory allows us to calculate forces due to one- and two-particle transfer, while three-particle and higher-multiplicity transfer must at this stage be treated phenomenologically. Let us consider some specific situations, remembering that the possible system to be transferred must obey all the conservation laws of strong interactions.

1. **Nucleon-nucleon scattering.** Here the longest-range force (or the nearest unphysical singularity) comes from single-pion transfer, while the next longest is due to two pions. Both of these are calculable, but forces of a range shorter than one-third of a pion Compton wavelength must await a

treatment of the many-body problem. Note that although we could attempt to calculate the force due to K-$\overline{K}$ transfer there is not much point in doing so because the mass of two kaons is as great as that of seven pions. On the other hand, if a hitherto undiscovered particle exists, of zero strangeness and mass less than three pions, then its contribution to the nuclear force should be experimentally identifiable.

2. **Pion-pion scattering.** Here all odd-pion transfers are forbidden, so the longest-range force is due to pion pairs, and the incomplete theory carries us down to one-quarter of a pion Compton wavelength in the force range.

3. **Pion-nucleon scattering.** Here there are two kinds of long-range forces. The "direct" forces arise from transfer of systems of zero baryon number, of which pion pairs are the least massive, and pion quartets the first configuration that must be treated phenomenologically. (Odd-pion transfer is again forbidden.) However, a very important "baryon exchange" force also must be considered because of the large difference in mass between pion and nucleon. That is to say, the original nucleon can "emit" a virtual nucleon, becoming a pion, with a violation of energy conservation that is determined not by the nucleon mass alone but rather (it turns out) by the geometric mean of nucleon and pion masses. This virtual nucleon moves across to the initial pion and is absorbed, transforming it into a nucleon and transmitting a force whose range turns out to be (by the uncertainty principle) approximately $(m_\pi m_N)^{-1/2}$, comparable to the range of 2π exchange. The incomplete theory can handle also the corresponding force when a single pion accompanies the exchanged nucleon, but more than one pion "fellow traveler" is beyond our powers at present. We shall see that the complicated nature of the pion-nucleon force is reflected in a complicated arrangement of singularities in the complex plane. By contrast, the singularities of the π-π and N-N amplitudes have a simple structure.

It is clear that because of their small mass, pions play a central role in this kind of approach to a theory of strong interactions. It will be impossible to go any distance without understanding the two-pion system, which occurs prominently not only in the long-range parts of the above-listed interactions but in many other processes as well. This circumstance alone would justify devoting much of our attention in these lectures to the two-pion configuration; another reason, however, is that, of all strongly interacting systems, the π-π is the easiest to handle, while at the same time it contains all the essential features of the general S-matrix approach. If one understands clearly what can be done and what cannot be done in the S-matrix framework for the π-π problem, a satisfactory foundation for discussing all strong-interaction problems will have been laid.

2 THE LORENTZ-INVARIANT ELASTIC AMPLITUDE AND THE SUBSTITUTION LAW

For reasons of convenience Möller,[12] in his 1945 paper clarifying the earlier proposals of Heisenberg,[2] introduced a matrix S − 1, and then in addition factored out an energy-momentum delta function, leaving a Lorentz-invariant function of n − 1 momentum variables, where n is the total number of particles involved (ingoing plus outgoing). This reduced matrix has been called by various names; we shall be mainly concerned with the case n = 4, where the reduced matrix element will simply be referred to as the "invariant amplitude." The normalization of the invariant amplitude has not been standardized; we shall choose it as close as possible to the "physical amplitude" $f(\theta)$, which is defined, except for a phase, by the barycentric-system differential cross-section formula

$$d\sigma/d\Omega = (q_f/q_i)|f(\theta)|^2 \tag{2-1}$$

Here q_f and q_i are the final and initial magnitudes of three momenta and θ is the angle between; q_i and q_f are of course equal for elastic scattering. The phase of $f(\theta)$ will be conventional; i.e., $f(\theta)$ becomes real as the interaction becomes weak, positive for attraction and negative for repulsion in the elastic case. More precisely, for elastic scattering of particles with zero spin,

$$f(\theta) = (1/q) \sum_{l=0}^{\infty} (2l + 1)\, e^{i\delta_l} \sin\delta_l\, P_l(\cos\theta) \tag{2-2}$$

where δ_l is the phase shift in the state of orbital angular momentum l. Möller showed that either for inelastic or elastic scattering the factor connecting f to the invariant amplitude for zero spin is simply W, the total

energy in the barycentric system.[12] Thus we normalize our invariant amplitude A by the formula

$$A = (W/2)f \tag{2-3}$$

The invariant-cross-section formula (for an arbitrary Lorentz frame) in terms of $|A|^2$ may be found in Möller's article, as well as in many textbooks. All we need here is the knowledge that A is Lorentz-invariant.

With zero spin, A can depend only on the invariants that may be formed from the three independent four-momenta remaining after energy-momentum conservation is applied. To maintain a maximum symmetry let us assign four-momenta, p_1, p_2, p_3, p_4, all of which correspond *formally* to ingoing particles. Two of these momenta will always be positive timelike, representing the actual ingoing particles, while the other two are negative timelike and represent the actual outgoing antiparticles. Energy-momentum conservation is stated through the condition

$$p_1 + p_2 + p_3 + p_4 = 0 \tag{2-4}$$

while the particle masses are introduced through the four constraints

$$p_i^2 = m_i^2 \tag{2-5}$$

It is convenient to define three invariants

$$\begin{aligned} s_1 &= (p_1 + p_4)^2 = (p_2 + p_3)^2 \\ s_2 &= (p_2 + p_4)^2 = (p_1 + p_3)^2 \\ s_3 &= (p_3 + p_4)^2 = (p_1 + p_2)^2 \end{aligned} \tag{2-6}$$

each of which is the square of the total energy in the barycentric system for a particular pairing of incoming and outgoing particles. With the constraints (2-4) and (2-5), s_1, s_2, and s_3 are not independent of one another but satisfy the relation

$$s_1 + s_2 + s_3 = m_1^2 + m_2^2 + m_3^2 + m_4^2 \tag{2-7}$$

Thus, any two of the s variables are to be considered as independent, with the third determined by (2-7). We now assert that our invariant amplitude A is a function only of the two independent s variables.

It is trivial to verify that no further independent scalars can be formed from $p_1 \cdots p_4$. That there are two and only two could have been anticipated by realizing that in the barycentric system the scattering depends on energy and angle and nothing more. What is not trivial, however, is to say that A cannot depend on which of the four-vectors p_i is positive timelike and which negative, i.e., on which particles are incoming and which outgoing. A Lor-

entz transformation cannot interchange positive and negative timelike vectors, so we are going beyond Lorentz invariance; we are invoking the notion of TCP invariance.

Consider the six reactions represented by Fig. 2-1. We can classify these by pairing the particles—two incoming and two outgoing—to define three "channels." Channel I is that channel for which s_1 is the square of the total energy in the barycentric system, pairing p_1 with p_4 and p_2 with p_3. It describes the reaction $\pi^- + \bar{\Lambda} \rightarrow K_0 + \bar{p}$ as well as the TCP equivalent antiparticle reaction†

$$\bar{K}_0 + p \rightarrow \pi^+ + \Lambda$$

Reactions with π^- ingoing have p_{10} positive, while those with π^+ outgoing have p_{10} negative. The signs of the energy components of the other four-momenta obey a similar rule.

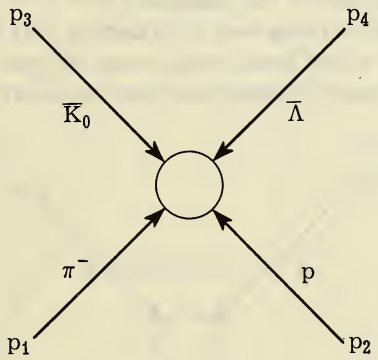

FIG. 2-1. Diagram describing the reactions

I. $\bar{K}_0 \rightarrow p \rightarrow \pi^+ + \Lambda$

II. $\pi^- + \bar{K}_0 \rightarrow \Lambda + \bar{p}$

III. $\pi^- + p \rightarrow K_0 + \Lambda$

as well as the corresponding antiparticle reactions.

Therefore Channel I is characterized by the fact that s_1 is positive and greater than some "threshold" value. Channel II is that for which s_2 is greater than some positive threshold, and includes the two reactions corresponding to the pairing (2,4) and (1,3), while Channel III makes the

†Because strong interactions have special symmetries, time reversal, charge conjugation, and charge independence, a single invariant amplitude actually can describe many more than the two TCP equivalent reactions. It is confusing, however, to invoke these additional symmetries before understanding the general features of the S-matrix approach.

final pairing (3,4) and (1,2) and of course has s_3 above threshold. It is possible to verify that the physical regions of the s variables for the different channels do not overlap. Consider the simple case in which all four masses are equal, e.g., the diagram of Fig. 2-2, which includes the three basic reactions, $n + p \rightarrow n + p$, $n + \bar{p} \rightarrow n + \bar{p}$, and $p + \bar{p} \rightarrow n + \bar{n}$. If q and θ are the barycentric-system three-momenta and angle for neutron-proton scattering, then

$$\begin{aligned} s_1 &= -2q^2(1 + \cos\theta) \\ s_2 &= -2q^2(1 - \cos\theta) \\ s_3 &= 4(q^2 + M^2) \end{aligned} \tag{2-8}$$

and we see that in the physical region of Channel III ($n + p \rightarrow n + p$), s_3 is positive and greater than $4M^2$, while s_1 and s_2 are both negative. Obviously, in the physical region for Channel II ($p + \bar{p} \rightarrow n + \bar{n}$), s_2 is greater than $4M^2$ while s_1 and s_3 are negative; for Channel I ($n + \bar{p} \rightarrow n + \bar{p}$) the positive variable is s_1. In general those two s variables that for a particular channel are not the square of the total energy may be interpreted as the negative squares of momentum transfer and have physical ranges that extend to minus infinity.

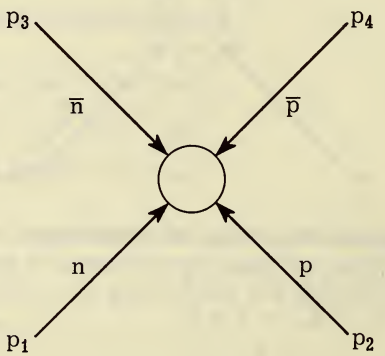

FIG. 2-2. Diagram for nucleon-nucleon and nucleon-antinucleon scattering:

I. $n + \bar{p} \rightarrow n + \bar{p}$

II. $p + \bar{p} \rightarrow n + \bar{n}$

III. $n + p \rightarrow n + p$

The foregoing situation is made even clearer by the use of a diagram suggested by Mandelstam[10] that depicts the three variables, s_1, s_2, s_3, in a symmetrical way on a two-dimensional plot. This diagram is shown in

Fig. 2-3 for the equal-mass case. The general prescription for constructing such a diagram has been given by Kibble.[13]

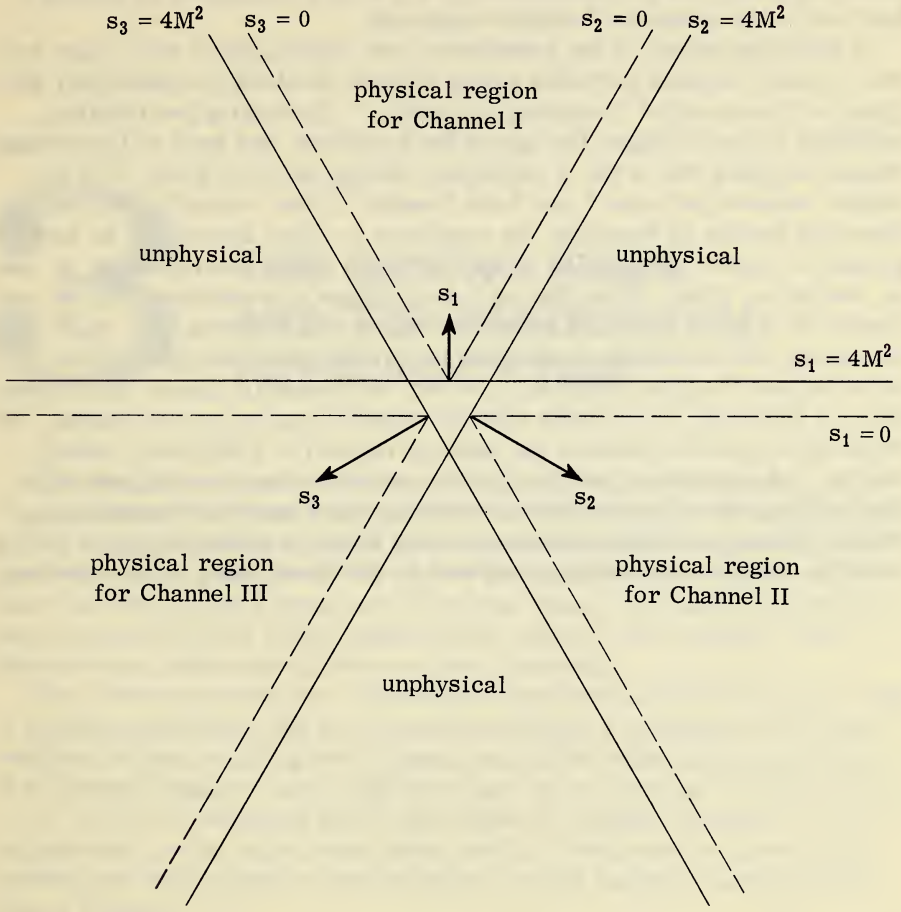

FIG. 2-3. The Mandelstam diagram for equal-mass elastic scattering.

We may now state the substitution law, which in the present framework takes a very simple form. We postulate that a single analytic function of two variables, $A(s_1, s_2)$, describes all three channels corresponding to a given diagram, the channel being selected merely by assigning the values of the variables. The key word here is "analytic." Since the ranges for the three channels do not overlap one must have a procedure of continuation to give such a postulate any physical content. In the diagram approach the substitution law is a direct consequence of the Feynman rules and has a clear meaning for any diagram whose singularities have been analyzed. All diagrams with four external lines analyzed to date have a singularity structure

permitting analytic continuation between the three physical regions. The substitution law has such a simple and plausible appearance that for many years its power was not appreciated, but it is now recognized as playing a key role in the dynamical S-matrix approach.

A principle related to the substitution law, which applies when there are two or more identical particles among the four involved in a particular diagram, is the so-called "crossing symmetry." Exchanging two identical particles at most changes the sign of the amplitude, and such an interchange means switching two of the s variables, leaving the third alone. For example, suppose particles 1 and 3 are identical. Then, depending on whether these are bosons or fermions, the amplitude is either symmetric or antisymmetric under exchange of p_1 and p_3, which means interchanging s_1 and s_3, leaving s_2 alone. [Note that such an exchange is consistent with the constraint (2-7).] If 1 and 3 are both incoming or both outgoing, i.e., $(s_2)^{1/2}$ is the energy, the symmetry in question is just the Pauli principle. If one is incoming and the other outgoing, however, the symmetry cannot be so identified. In this case, if one starts with the physical region for one channel, the exchange in question leads to the physical region for a different channel. (In Fig. 2-3 crossing is just a reflection about the line bisecting one of the physical regions.) Thus crossing symmetry has a general meaning only when continuation of the amplitude between different physical regions is possible. Such a continuation is allowed by the Mandelstam representation.

3 THE LANDAU RULES AND THE MANDELSTAM REPRESENTATION

The Mandelstam representation is a prescription for the location of the singularities of an S-matrix element for two incoming and two outgoing particles. It is a special case of the more-general prescription given by Landau. Neither the general nor the special prescription can be said to have been "derived" from a clear set of physical ideas. The basis at present is partly plausibility and partly experimental success; no violations of the Mandelstam-Landau principles have been observed.

The closest approach to a "derivation" has been achieved through a study of Feynman diagrams, and it is in such terms that a statement of the prescription is most easily given. Landau was able to show that singularities of a Feynman diagram occur only for values of the external variables (e.g., s_1, s_2, s_3 in the discussion above) that allow all internal momenta to be simultaneously on the appropriate mass shells.[4] Furthermore, these singularities are either poles or branch points, so that analytic continuation is always possible..

To illustrate the Landau principle, let us refer to the S-matrix element of Fig. 3-1, which describes reactions with two nucleons and two pions. In order to deduce the location of the singularities of the invariant amplitude $A(s_1, s_2, s_3)$, we are supposed to consider all possible Feynman graphs with external lines corresponding to Fig. 3-1. A simple subset of graphs is indicated in Fig. 3-2 (with the directions of two lines reversed). We include in this subset all graphs that have a single "virtual" nucleon line standing between a "blob" where π° and p are absorbed and a "blob" where π^+ and n are emitted. These blobs are supposed to contain all possible complications, so that an infinite number of Feynman graphs is being considered. Landau's rule tells us immediately that all these graphs contain a simple pole at $s_1 = M^2$, since this isolated point in the complex s_1 plane is selected

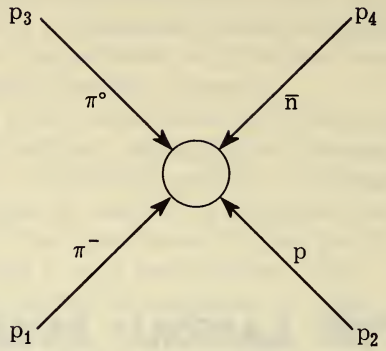

FIG. 3-1 Diagram for the three channels

I. $\pi^\circ + p \rightarrow \pi^+ + n$

II. $\pi^- + \pi^\circ \rightarrow n + \bar{p}$

III. $\pi^- + p \rightarrow \pi^\circ + n$

by energy-momentum conservation when the intermediate proton is put on its mass shell. How does such a pole arise in conventional renormalization theory?

The sum of all graphs of the type of Fig. 3-2 may be written

$$F_{\pi^\circ p}{}^{p}(s_1)\, S^{p}(s_1)\, F_{\pi^+ n}{}^{p}(s_1) \tag{3-1}$$

where the F are vertex functions on the mass shell for the two particles indicated in the subscript but considered as a function of the square of the mass of the intermediate proton (which here is equal to s_1, by momentum-

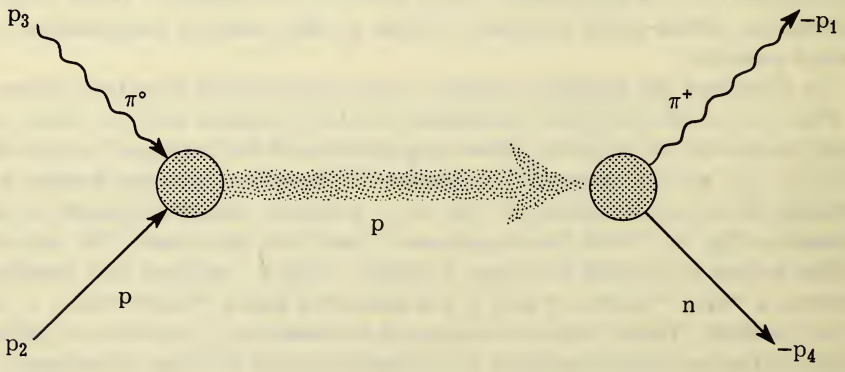

FIG. 3-2. Subset of Feynman graphs with a single intermediate proton.

energy conservation). Here S^p is the renormalized proton propagator and it is well known that $S^p(s_1)$ has a pole of unit residue at $s_1 = M^2$.[14] It is also known that the vertex functions are analytic in the neighborhood of this point.[14] Thus we see that the prescription of putting the intermediate proton on its mass shell correctly gives this simplest type of singularity.

Previous studies of the propagation and vertex functions occurring in (3-1) have shown that they are in general real analytic functions with branch cuts extending from $(M + 1)^2$ to infinity.[14] The product (3-1) may thus be represented through the Cauchy theorem as

$$R/(M^2 - s_1) + \int_{(M+1)^2}^{\infty} ds_1' \rho(s_1')/(s_1' - s_1) \qquad (3\text{-}2)$$

where R and ρ are real.†

The branch point at $s_1 = (M + 1)^2$ is predicted by Landau's rule when diagrams with a nucleon *and* a pion somewhere between the two "blobs" are considered. However, many such diagrams, a simple example of which is shown in Fig. 3-3, require a simultaneous consideration of the other variables, s_2 and s_3. Mandelstam[10] first analyzed the singularity structure in

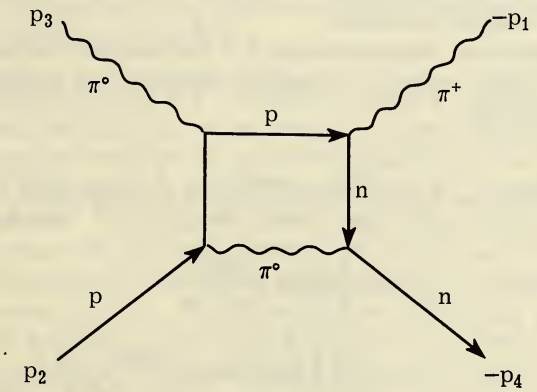

FIG. 3-3. A fourth-order "box" diagram.

two independent complex variables of fourth-order "box" diagrams and showed that the graph of Fig. 3-3 can be represented as

$$\int_{}^{\infty}\int_{}^{\infty} \frac{\rho_{12}(s_1', s_2')}{(s_1' - s_1)(s_2' - s_2)}\, ds_1' ds_2' \qquad (3\text{-}3)$$

†We shall often use the pion mass as a unit, neglecting the difference between the masses of charged and neutral pions. We always take $\hbar = c = 1$.

where ρ_{12} is real and the two-dimensional integration covers that region of s_1' and s_2' for which the internal momenta of Fig. 3-3 can be on the appropriate mass shells. The solution of this essentially geometrical problem is shown in Fig. 3-4. The calculation of the curved boundary will be considered later in detail; for the moment it is sufficient to say that the boundary asymptotically approaches $(2M)^2$ in s_2 and $(M + 1)^2$ in s_1. These are the minimum masses possible for a two-nucleon or a pion-nucleon system, respectively. They determine the branch points of (3-3) when one of the two variables s_1 and s_2 is held fixed and the other extended into the complex plane.

Mandelstam conjectured that in fact all Feynman graphs associated with the process of Fig. 3-1 could be represented by formulas of the type (3-2) with appropriate regions of integration. The conjecture has been verified in many special cases and no counter examples have been found; Eden, Landshoff, Polkinghorne, and Taylor have gone a considerable distance toward a general verification for an arbitrary graph.[15] We shall assume here that the conjecture is correct for the particular two-body problems to be studied, but it should be realized that the assumption is not crucial. As long as one knows the location of all singularities, and, with sufficient patience, this information is always provided by Landau's rule,† then one may proceed with the dynamical S-matrix program.

The full Mandelstam representation for the invariant amplitude $A(s_1, s_2, s_3)$ is then as follows, except for possible subtractions needed if the integrals do not converge:

$$
\begin{aligned}
A(s_1, s_2, s_3) = {} & \frac{1}{\pi}\int ds_1' \frac{\rho_1(s_1')}{s_1' - s_1} + \frac{1}{\pi}\int ds_2' \frac{\rho_2(s_2')}{s_2' - s_2} \\
& + \frac{1}{\pi}\int ds_3' \frac{\rho_3(s_3')}{s_3' - s_3} \\
& + \frac{1}{\pi^2}\int\int ds_1' ds_2' \frac{\rho_{12}(s_1', s_2')}{(s_1' - s_1)(s_2' - s_2)} \\
& + \frac{1}{\pi^2}\int\int ds_1' ds_3' \frac{\rho_{13}(s_1', s_3')}{(s_1' - s_1)(s_3' - s_3)} \\
& + \frac{1}{\pi^2}\int\int ds_2' ds_3' \frac{\rho_{23}(s_2', s_3')}{(s_2' - s_2)(s_3' - s_3)}
\end{aligned}
\tag{3-4}
$$

†For certain mass ratios one encounters anomalous singularities, that is, branch points not determined simply by the masses of intermediate states. See, e.g., R. E. Cutkosky, Revs. Modern Phys., **33**, 448 (1961). All the problems discussed in these lectures are believed to have normal singularities only.

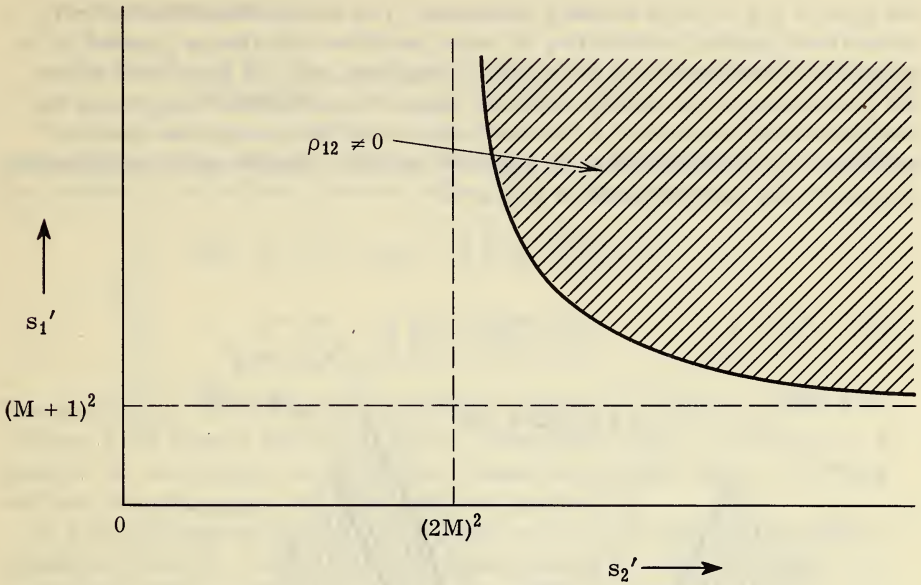

FIG. 3-4. Boundaries of the double-spectral function for the "box" diagram of Fig. 3-3.

where the integrations in each s′ variable go over a region of the positive real axis corresponding to the mass squared of actual physical systems that have the quantum numbers of the corresponding channel.

For example: Channel I in Fig. 3-1 has charge +1, baryon number +1, and zero strangeness. The lightest system with these quantum numbers is the proton with mass M; the next lightest are (π°, p) and (π^+, n), with a range of masses starting from M + 1 and extending to infinity. More massive systems, containing 3,4,..., etc., particles, fall in this range, so we conclude that the spectrum $\rho_1(s_1')$ has a "line" at $s_1' = M^2$ and a "continuum" for $(M+1)^2 < s_1' < \infty$. The "line" obviously leads to the pole already discussed and once this is removed the lower limit on the ds_1' integration is equal to $(M+1)^2$.

For Channel II there is no pole because we know of no single particle that has the same quantum numbers as two pions, i.e., zero baryon number, zero strangeness, and parity $(-1)^J$. The continuum starts here with the two-pion system at $s_2' = 4$; the $\overline{N}N$ threshold at $s_2' = 4M^2$ is much higher. Channel III is similar to Channel I; in fact when the notion of charge independence is introduced we shall be able to relate these two channels by crossing symmetry.

The one-dimensional integrals in (3-4) correspond to Feynman diagrams in which, at some point, a single-particle line joins the initial and final configurations. That is, the integral containing ρ_1 represents all diagrams of

the type of Fig. 3-2, as already discussed. The double integrals in (3-4) arise from graphs in which two or more particles are always present in intermediate configurations for *all three* channels, and, as mentioned above, the boundaries of the double-spectral regions asymptotically approach the smallest mass squared of a two-body state with the appropriate quantum numbers. As an illustration, in Fig. 3-5 the three double-spectral regions are shown for the amplitude of Fig. 2-2.

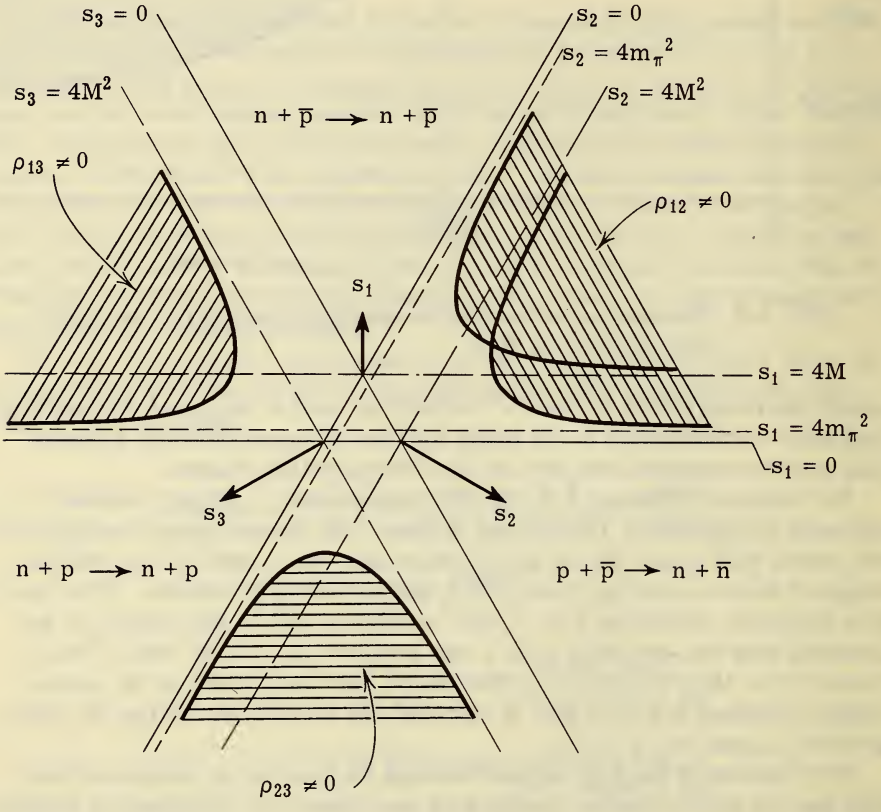

FIG. 3-5. The double-spectral regions for the amplitude of Fig. 2-2.

For a variety of reasons it is useful to exhibit the connection between the Mandelstam representation (3-4) and one-dimensional dispersion relations. First observe that in the physical region for Channel I, say, the only denominators that vanish in formula (3-4) are those containing the factor

$s_1' - s_1$. Remembering

$$\frac{1}{s_1' - s_1 \mp i\varepsilon} = P\frac{1}{s_1' - s_1} \pm i\pi\delta(s_1' - s_1) \tag{3-5}$$

we can then easily calculate the discontinuity of A in this region, crossing the positive s_1 axis from above to below, to be $2iA_1$, where

$$A_1(s_1, s_2, s_3) = \rho_1(s_1) + \frac{1}{\pi}\int ds_2' \frac{\rho_{12}(s_1, s_2')}{s_2' - s_2} + \frac{1}{\pi}\int ds_3' \frac{\rho_{13}(s_1, s_3')}{s_3' - s_3} \tag{3-6}$$

We shall define A_1 by formula (3-6) even outside the physical region for Channel I and always refer to it as the "absorptive part" for Channel I. In general, A_1 is complex but not in the Channel I physical region; here it is real and in fact is precisely the imaginary part of A.

In a similar way we can define A_2 and A_3 to be the absorptive parts for Channels II and III, respectively. It is then simply a matter of algebra to verify that (3-4) can be written in three possible ways in terms of A_1, A_2, A_3:

$$A(s_1, s_2, s_3) = \frac{1}{\pi}\int ds_1' \frac{\rho_1(s_1')}{s_1' - s_1} + \frac{1}{\pi}\int ds_2' \frac{A_2(s_1, s_2', \Sigma m^2 - s_1 - s_2')}{s_2' - s_2} + \frac{1}{\pi}\int ds_3' \frac{A_3(s_1, \Sigma m^2 - s_1 - s_3', s_3')}{s_3' - s_3} \tag{3-7a}$$

$$A(s_1, s_2, s_3) = \frac{1}{\pi}\int ds_2' \frac{\rho_2(s_2')}{s_2' - s_2} + \frac{1}{\pi}\int ds_1' \frac{A_1(s_1', s_2, \Sigma m^2 - s_1' - s_2)}{s_1' - s_1} + \frac{1}{\pi}\int ds_3' \frac{A_3(\Sigma m^2 - s_2 - s_3', s_2, s_3')}{s_3' - s_3} \tag{3-7b}$$

$$A(s_1, s_2, s_3) = \frac{1}{\pi}\int ds_3' \frac{\rho_3(s_3')}{s_3' - s_3} + \frac{1}{\pi}\int ds_1' \frac{A_1(s_1', \Sigma m^2 - s_1' - s_3, s_3)}{s_1' - s_1} + \frac{1}{\pi}\int ds_2' \frac{A_2(\Sigma m^2 - s_2' - s_3, s_2', s_3)}{s_2' - s_2} \tag{3-7c}$$

The first form is usually called the one-dimensional dispersion relation for fixed s_1, the second for fixed s_2, and the third for fixed s_3. In our pion-nucleon example the form that has received the most attention is that for s_2 fixed at a negative value in its momentum-transfer range, where the two absorptive parts that occur both correspond to pion-nucleon scattering in a more or less physical region (see the Appendix). In what follows, however, we shall often find it necessary to put the fixed variable into its positive (energy) range; the absorptive parts then are nonphysical and have a meaning only through (3-6).

4 CUTKOSKY'S GENERALIZED UNITARITY RELATION

We now come to a question of obviously dynamical content. How does one evaluate the residues of the poles and the discontinuities across the cuts—or in other words—the single- and double-spectral functions in the Mandelstam representation? The residues of the poles are fairly straightforward. For example, the proton pole in s_1 discussed in connection with Fig. 3-2 has a residue according to (3-1) that is equal to

$$F_{\pi^0 p}{}^{p}(M^2)\, F_{\pi^+ n}{}^{p}(M^2) = g_{op} g_c \qquad (4\text{-}1)$$

since the renormalized coupling constants g_{op} and g_c are defined as the values of the appropriate pion-nucleon vertex functions with all three particles on the mass shell. It is evident that these considerations apply to any pole that may occur, always leading to the residue as a product of two vertex functions each on the mass shell. Note that the residues are always real.

It is possible to deduce the above recipe for relating poles in scattering amplitudes to coupling constants without reference to Feynman diagrams, but the machinery is cumbersome. Particularly when spin is present it is a great convenience to know that the Feynman rules, applied to diagrams of lowest order, lead to the correct connection between residues and conventional coupling constants. This fact of course explains why perturbation theory, blindly applied, occasionally gives sensible answers even in strong-interaction problems. Poles dominate the behavior of the scattering amplitude in their immediate neighborhood, so if one happens to be discussing experiments close to a pole, the lowest-order perturbation formula may be reliable.

The generalization of the above recipe, suitable for the discontinuity across branch cuts, has been given by Cutkosky,[5] who showed that for an arbitrary (reduced) graph one simply replaces internal propagators

$(M^2 - p^2)^{-1}$ by delta functions $i\pi\delta(M^2 - p^2)$. It is trivial that this recipe correctly gives the residues of poles; it is also easy to see that it contains the ordinary unitarity condition. Unitarity states that $S^+S = 1$, and if we write $S = 1 + 2iT$, we find the equivalent relation

$$\text{Im } T_{ba} = \sum_n T_{nb}^* T_{na} \tag{4-2}$$

Now for a reaction with two particles, p_1 and p_2, in, and two particles, $-p_3$ and $-p_4$, out, we have†

$$T_{ba} \sim A(s_1, s_2, s_3)$$

Thus (4-2) tells us that in the physical region for Channel III,

$$\text{Im } A(s_1, s_2, s_3) \sim \sum_n \langle n | T | -p_3, -p_4 \rangle^* \langle n | T | p_1, p_2 \rangle \tag{4-3}$$

where the states n all have a "mass" equal to $(s_3)^{1/2}$. Now refer to the reduced graph of Fig. 4-1, which, according to Cutkosky, gives the discontinuity in A associated with s_3. Evidently, if all the internal lines are put on the appropriate mass shells, the result is closely connected with the right-hand side of Eq. (4-3), and, in fact, if careful track is kept of all factors, one finds exactly this sum over states. In the previous section we saw that the discontinuity in A in a physical region is essentially Im A; thus the Cutkosky rule leads to (4-3) and may be appropriately called a generalized unitarity relation.

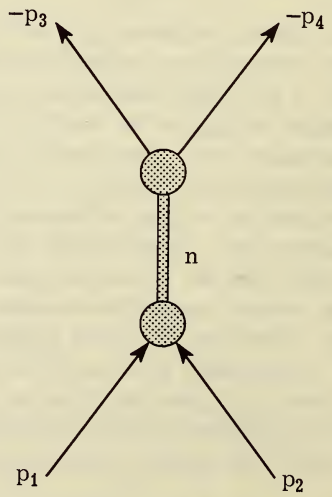

FIG. 4-1. The reduced (Cutkosky) diagram giving the discontinuity associated with s_3.

†In this section, all kinematical factors are omitted.

The rule goes beyond ordinary unitarity, however, in that it gives the discontinuities also in unphysical regions, for example, the double-spectral regions of the Mandelstam representation. Cutkosky showed that in order to calculate the Mandelstam double-spectral functions one considers all reduced graphs with four vertices and replaces propagators by mass-shell delta functions. When applied to the box diagram of Fig. 4-2, this rule gives

$$\rho_{13}(s_1,s_3) \sim \int d^4k\,\delta(m_1^2 - k^2)\delta[m_2^2 - (k + p_2)^2] \times \delta[m_3^2 - (k + p_2 + p_3)^2]\delta[m_4^2 - (k + p_2 + p_3 + p_4)^2] \tag{4-4}$$

where $m_1 \cdots m_4$ are the masses of the four internal particles. The integral can be carried out explicitly and for the case of equal-mass external particles with two of the internal masses, m_2 and m_4, equal to the external mass M and the other two equal to m, one gets a particularly simple result. The double-spectral function is given by

$$\rho_{13}(s_1,s_3) \sim 1/(s_1s_3)^{1/2}\ 1/[(s_3 - 4M^2)(s_1 - 4m^2) - 4m^4]^{1/2} \tag{4-5}$$

for the argument of the square bracket positive, and $\rho_{13} = 0$ otherwise. The boundary of the double-spectral region is thus determined by the curve

$$(s_3 - 4M^2)(s_1 - 4m^2) = 4m^4 \tag{4-6}$$

in agreement with our previous qualitative remarks and as shown explicitly in Fig. 3-5.

The generalization of (4-4) to more-complicated four-vertex graphs is evident. For example, an important class is that of Fig. 4-3, which contains

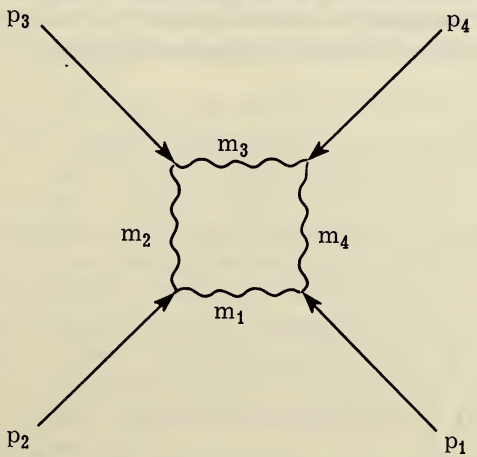

FIG. 4-2. The general box diagram.

only two intermediate particles in Channel III but an arbitrary number (and variety) in Channel I. Cutkosky pointed out[5] that the sum of all such graphs can be written as an integral over a product of two absorptive parts A_i, of the type defined above, a circumstance that can be understood in terms of formula (4-3). Later we shall make essential use of this strikingly simple result.

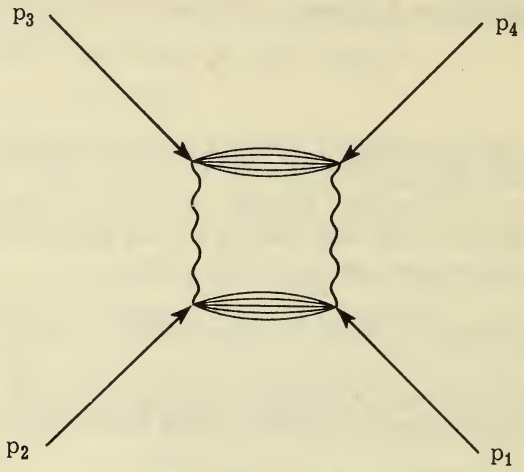

FIG. 4-3. Cutkosky graph for ρ_{13} containing two particles in Channel III and an arbitrary number in Channel I.

In general, Cutkosky tells us that the single- and double-spectral functions are determined by integrals over products of various S-matrix elements, always on the mass shell although often in unphysical regions. We shall see below that such a prescription in principle completes the dynamical scheme, yielding a set of coupled integral equations.

5 GENERALIZATION TO INCLUDE CHARGE AND SPIN

The possibility of degrees of freedom of charge and spin has so far been ignored. However, internal degrees of freedom may always be absorbed into invariant matrices, whose coefficients are invariant functions of the s variables only. The number of such functions depends on the complexity of the internal degrees of freedom, and generally the vector addition rule can be used in counting. For example, the pion has zero spin and isotopic spin 1, while the nucleon has spin 1/2 and isotopic spin 1/2. The different possible total I-spin values for a 2π system are 0, 1, and 2, so that we expect three independent invariant π-π amplitudes. The possible total I-spin values for a π-N system are 1/2 and 3/2, but in addition the nucleon spin can combine in two ways with any given orbital angular momentum l to form $J = l + 1/2$ or $J = l - 1/2$. Thus there are four independent invariant π-N amplitudes. The N-N system is even more complicated; if one counts carefully here, the result is ten.

A proper choice of invariant spin matrices leads to invariant amplitudes with exactly the same singularity structure as expressed by Eq. (3-4) for the zero-spin case. The choice of charge matrices is obviously irrelevant to analyticity properties, although certain choices may be more convenient than others for calculation. To obtain correct spin matrices, a possible procedure is to guess the answer and then check. This has worked so far in all cases involving particles of spin zero and spin 1/2, as well as photons. The procedure has been described in some detail for the nontrivial N-N case by Goldberger, Grisaru, McDowell, and Wong.[16] Recently a systematic approach has been developed by Hearn.[17] We shall not delve into the spin problem here but simply state for illustration the well-known charge-spin result for the π-N system.[18]

For a diagram of the type of Fig. 3-1, rather than specifying the charge, let us label the pion lines each by the conventional isotopic vector index that

takes on the values 1, 2, 3. With the pion of momentum p_1 associate the index α, and with p_3 the index β. Nucleon-charge degrees of freedom may be suppressed into initial and final isospinors, and the problem is then to form two charge-independent combinations of nucleon isotopic-spin operators, τ_k. A possible choice is $1/2(\tau_\beta\tau_\alpha + \tau_\alpha\tau_\beta) = \delta_{\beta\alpha}$ and $1/2(\tau_\beta, \tau_\alpha)$, the one symmetric and the other antisymmetric under pion exchange. Nucleon spin may be similarly suppressed into initial and final (four-component) spinors, and a choice made of two independent Lorentz invariants constructed from the Dirac matrices γ_μ and the four-momenta. Here the correct choice is essentially unique and turns out to be 1 and $(1/2)i\gamma\cdot(p_1 - p_3)$. A linear combination of these matrices with *constant* coefficients is of course satisfactory, but s polynomials must be avoided in the coefficients or, as explained below, extra poles may be produced in the invariant amplitudes.

The complete amplitude for a π-N diagram of the type of Fig. 3-1 may be written

$$\bar{u}_{-p_4}\left\{\delta_{\beta\alpha}\left[-A^0(s_1, s_2, s_3) + \frac{1}{2}i\gamma\cdot(p_1 - p_3)\,B^0(s_1, s_2, s_3)\right]\right.$$
$$+\frac{1}{2}(\tau_\beta, \tau_\alpha)\left[-A^1(s_1, s_2, s_3)\right.$$
$$\left.\left.+\frac{1}{2}i\gamma\cdot(p_1 - p_3)\,B^1(s_1, s_2, s_3)\right]\right\}u_{p_2} \qquad (5\text{-}1)$$

and the connection with amplitudes for well-defined total I spin is easily obtained. For example, for Channel III,

$$\pi(p_1, \alpha) + N(p_2) \rightarrow \pi(-p_3, \beta) + N(-p_4)$$

we find for the two values $I = 1/2, 3/2$,

$$A_{III}^{1/2} = A^0 + 2A^1 \qquad B_{III}^{1/2} = B^0 + 2B^1$$
$$A_{III}^{3/2} = A^0 - A^1 \qquad B_{III}^{3/2} = B^0 - B^1$$

On the other hand, for Channel II,

$$\pi(p_1, \alpha) + \pi(p_3, \beta) \rightarrow N(-p_2) + N(-p_4)$$

we find that except for normalization the amplitudes for the two total isotopic spin values 0, 1 are just the quantities already labeled with these superscripts.[19] Our particular choice of matrices above is motivated by crossing symmetry. Under interchange of the two pions, $\alpha \leftrightarrow \beta$ and $p_1 \leftrightarrow p_3$, so that A^0 and B^1 are symmetric under interchange of s_1 and s_3 while A^1 and B^0 are antisymmetric.

It can be verified that the four invariant functions $A^{0,1}$, $B^{0,1}$ satisfy the Mandelstam representation—that is, they have only "dynamical" singularities arising from intermediate states in the various channels—if one accepts the corresponding conjecture for zero spin. However, had we used an essentially different choice of matrices, e.g., 1 and $\gamma \cdot p_1 \gamma \cdot p_2$, we should have found additional (kinematical) poles in the corresponding invariant functions. Care must therefore be used in setting up problems with spin, but once a correct choice of amplitudes has been made, the use of unitarity and analyticity to determine the S matrix follows the same line as that for zero spin.

6 PHYSICAL INTERPRETATION OF SINGULARITIES

We are now in a position to see in detail the connection between forces and singularities that was emphasized in the introduction. Singularities in the Mandelstam representation occur only when one or more of the denominators in (3-4) vanish, and this in turn happens only when an s variable is equal to the square of the mass of a strongly interacting physical system having the quantum numbers of the associated channel. If we are focusing attention on one particular channel, e.g., Channel III of Fig. 2-2 ($n + p \rightarrow n + p$), then we shall refer to the singularities "belonging" to this channel—that is, due to the vanishing of $s_3' - s_3$ denominators—as "physical" singularities. Most of these occur in the actual physical region of s_3 for Channel III, although there may be some extending for a distance below the true physical threshold. These "physical" singularities may be thought of as consequences of unitarity for Channel III, in contrast to the "unphysical" singularities associated with Channels I and II, which may be thought of as the "forces" giving rise to the Channel III reaction.

To bring out these ideas in a familiar situation, let us study formula (3-7c) as applied to the amplitude for Fig. 2-2, where we use (2-8) to replace s_1, s_2, s_3 by q^2 and $\cos\theta$, the barycentric-system variables for n-p scattering. Still ignoring nucleon spin and not worrying about a possible n-p bound state, we have

$$A(q^2,\cos\theta) = \frac{1}{\pi}\int_0^\infty dq'^2 \frac{\rho_3(q'^2)}{q'^2 - q^2} + \frac{1}{\pi}\int ds_1' \frac{A_1(s_1', 4M^2 - s_1' - s_3, s_3)}{s_1' + 2q^2(1 + \cos\theta)} + \frac{1}{\pi}\int ds_2' \frac{A_2(4M^2 - s_2' - s_3, s_2', s_3)}{s_2' + 2q^2(1 - \cos\theta)} \qquad (6\text{-}1)$$

One of the important features of this expression is that the $\cos\theta$ depend-

ence of the n-p amplitude comes entirely from the denominators of the last two terms. Let us compare the form (6-1) to the Born approximation for scattering by a Yukawa potential of range r_0. This is, familiarly,

$$\text{constant}/[(1/r_0^2) + 2q^2(1 - \cos\theta)] \tag{6-2}$$

Thus we see that the last term of (6-1) looks like the Born scattering due to a superposition of Yukawa potentials, where the range is $1/(s_2')^{1/2}$. The second term in (6-1) can similarly be identified with an exchange potential, but the first term is of a different type. This contributes only to S-wave n-p scattering, and qualitatively may be associated with the failure of the Born approximation when the S-phase shift is large. If some higher phase shifts also are large, further terms may be needed in (6-1), corresponding to subtractions in the original Mandelstam representation. We shall return to this subtle question in a later section.

For sufficiently high angular momentum, however, it is reasonable to expect the phase shift to be small when q^2 is small, so that the rough interpretation of the second two terms as the Fourier transforms of "potentials" is plausible. Now let us consider the strength and range of the "potentials," as controlled by the functions A_1 and A_2. The longest-range forces will come from the lowest values of s_1' and s_2', and these in turn are determined by the lowest masses of physical systems with the quantum numbers of Channel I and Channel II, respectively. Consider Channel I ($n + \bar{p} \rightarrow n + \bar{p}$), which gives us our exchange forces, and refer to formula (3-6) to see the structure of A_1. The term $\rho_1(s_1)$ contains a delta function corresponding to a discrete π^- state, and the coefficient of the delta function is g_c^2. Thus the longest-range exchange force has a range of one-pion Compton wavelength and a strength determined by the pion-nucleon coupling constant.

The next-longest-range force comes from the two-pion parts of the ρ_1, ρ_{12}, and ρ_{13} spectra. To achieve some understanding of this circumstance, we recall the unitarity condition (4-3) but applied now in the physical region for Channel I:

$$A_1 \sim \Sigma_m \langle m \mid T \mid n(-p_3), \bar{p}(-p_2)\rangle^* \langle m \mid T \mid n(p_1), \bar{p}(p_4)\rangle \tag{6-3}$$

It is possible to extend this formula to unphysical regions, and since all the elements of T conserve energy and momentum, we see that it is the unitarity condition which makes A_1 vanish except for values of $s_1 = (p_1 + p_4)^2$ equal to the squares of masses of systems that can be reached both from the initial and the final states of Channel I. For the range $4 < s_1 < 9$, only 2π states contribute to the sum in (6-3), so if we have some means of calculating the matrix element connecting nucleon-antinucleon states to two-pion states, we can calculate the "strength" of the exchange force for ranges between one-half and one-third of a pion Compton wavelength. Since this

matrix element corresponds to a four-line diagram (it is in fact Channel II of Fig. 3-1), there is hope that the calculation can be performed.

Formula (6-3) tells us how to calculate shorter-range forces due to multiparticle exchange if the matrix elements connecting these states to the $n\text{-}\bar{p}$ system are known. We do not yet have a definite method for calculating multiparticle matrix elements, but we observe that they are bounded in magnitude because of unitarity and therefore there is a limit to the possible strength of the forces that they generate.

Although we shall develop a more systematic approach later, formula (6-1) is actually suitable as it stands for calculating the high-angular-momentum parts of the n-p scattering amplitude, which are determined by the lower range of s_1' and s_2'. The modifications to take account of spin and charge have been worked out by several authors and expressions have been given for the one- and two-pion parts of A_1 and A_2.[20] The *very* high l-phase shifts are of course controlled by the one-pion parts alone, and thus by the pion mass and the pion-nucleon coupling constant, quantities already known with good accuracy. This circumstance has been exploited in recent phase-shift analyses of nucleon-nucleon scattering.[20]

It should be obvious that the considerations in this section are general. The forces producing a certain reaction are due to the intermediate states that occur in the two "crossed" reactions belonging to the same diagram. The range of a given part of the force is determined by the mass of the intermediate state producing it, and the strength of the force by the matrix elements connecting that state to the initial and final states of the crossed reaction. By considering all three channels on this basis we have a self-determining situation. One channel provides forces for the other two—which in turn generate the first. Our task now is to understand how many arbitrary parameters there are in such a situation and how to formulate a sensible method of calculation.

7 THE TWO-BODY DYNAMICAL EQUATIONS: DEFINITION OF THE "POTENTIAL"†

To achieve some orientation in the dynamics of a two-body problem as determined by unitarity and analyticity, let us consider for a moment nonrelativistic scattering by a superposition of Yukawa potentials:

$$V(r) = -\frac{1}{2M^2}\int_{m_0^2}^{\infty} dm^2 \rho_0(m^2)\frac{e^{-mr}}{r}$$

The scattering amplitude here (we may consider the ordinary amplitude, since the factor w is a constant, nonrelativistically) has been shown to satisfy a simple Mandelstam representation[22]:

$$f(q^2,t) = \frac{1}{\pi}\int_{t_0}^{\infty} dt' \frac{\rho_0(t')}{t'-t} + \frac{1}{\pi^2}\int\int dq'^2 dt' \frac{\rho(q'^2,t')}{(q'^2-q^2)(t'-t)} \qquad (7\text{-}1)$$

The single integral is essentially the Fourier transform of the potential (the Born approximation), with $t_0^{-1/2}$ determining the "range." The boundaries of the double-spectral function are as shown in Fig. 7-1 and are determined by the equivalent of formula (4-6), namely,

$$q^2(t-4t_0) = t_0^2 \qquad (7\text{-}2)$$

†The point of view expressed in this and the following chapter was developed by the author in collaboration with Frautschi,[21] but the underlying framework is due to Mandelstam.[10]

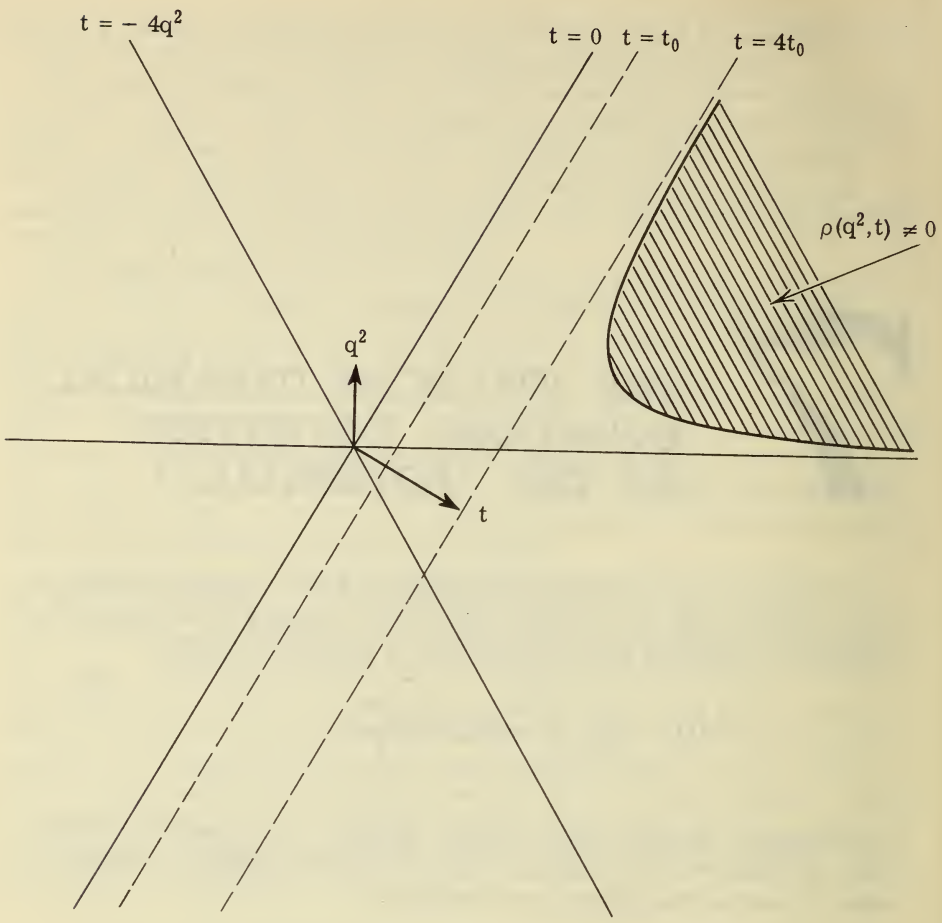

FIG. 7-1. Mandelstam diagram for potential scattering.

There is only one physical region, of course, and only one double-spectral region (with an exchange potential there is a second double-spectral region on the left side), but there are nevertheless many useful points of analogy with the general relativistic problem.

In particular one may ask how the double-spectral function is to be calculated from the unitarity condition. In this simple case the unitarity condition (4-3) becomes

$$\text{Im } f(q^2, \cos\theta) = \frac{q}{4\pi}\int d\Omega' \, f^*(q^2, \cos\theta') \, f(q^2, \cos(\theta,\theta'))$$

$$q^2 > 0, \quad -4q^2 < t < 0 \qquad (7\text{-}3)$$

where

$$\cos\theta = 1 + (t/2q^2) \tag{7-4}$$

$$d\Omega' = \sin\theta'\, d\theta'\, d\varphi' \tag{7-5}$$

and

$$\cos(\theta,\theta') = \cos\theta\ \cos\theta' + \sin\theta\ \sin\theta'\ \cos\varphi' \tag{7-6}$$

Let us define the imaginary (absorptive) part in the physical region as $f_3(q^2,t)$ and calculate from (7-1):

$$f_3(q^2,t) = \frac{1}{\pi} \int_{4t_0 + t_0^2/q^2}^{} dt' \frac{\rho(q^2,t')}{t' - t} \tag{7-7}$$

Thus if the Mandelstam representation (7-1) is correct, $f_3(q^2,t)$ is an analytic function of t with a cut along the positive t axis, and the discontinuity across this cut is the double-spectral function. Mandelstam was able to show that the unitarity condition (7-3) can be extended throughout the t complex plane in a manner consistent with the above-described situation. At the same time he obtained a formula for determining $\rho(q^2,t)$.[10]

The first step in the derivation is to define a function $f_2(q^2,t)$ which would be the absorptive part in the t channel—if there were a t channel,

$$f_2(t,q^2) = \rho_0(t) + \frac{1}{\pi} \int_{t_0^2/t - 4t_0}^{} dq'^2 \frac{\rho(q'^2,t)}{q'^2 - q^2} \tag{7-8}$$

and then to write

$$f(q^2,t) = \frac{1}{\pi} \int_{t_0}^{\infty} dt' \frac{f_2(t',q^2)}{t' - t} \tag{7-9}$$

as in formulas (3-7). Remembering (7-4) one then substitutes (7-9) into the right-hand side of (7-3) and carries out the angular integrations over $d\Omega'$ to find

$$f_3(q^2,t) = \frac{1}{4\pi^2 q} \int_{t_0}^{\infty} dt' \int_{t_0}^{\infty} dt''\, f_2^*(t',q^2)\, f_2(t'',q^2) \frac{1}{K^{1/2}(q^2;t,t',t'')} \times \ln \frac{t - t' - t'' - (t't''/2q^2) + (K)^{1/2}}{t - t' - t'' - (t't''/2q^2) - (K)^{1/2}} \tag{7-10}$$

where

$$K(q^2;t,t',t'') = t^2 + t'^2 + t''^2 - 2(tt' + tt'' + t't'') - (tt't''/q^2) \tag{7-11}$$

In (7-10) the branch of the logarithm to be taken is the one that is real in the physical region $-4q^2 < t < 0$. Examination of the right-hand side of (7-10) then reveals that for fixed values of t' and t'' the integrand is a real analytic function of t with a branch point at the zero of K for the larger value of t and an associated cut running along the positive real axis to infinity. The imaginary part of the logarithm along the cut turns out to be 2π, so we find

$$\rho(q^2,t) = \operatorname{Im} f_3(q^2,t) = \frac{1}{2\pi q}\int_{t_0} dt' \int_{t_0} dt'' \frac{f_2^*(t',q^2)\, f_2(t'',q^2)}{K^{1/2}(q^2;t,t',t'')} \qquad (7\text{-}12)$$

where the range of the dt' and dt'' integrals is restricted to those values such that

$$t > t' + t'' + \frac{t't''}{2q^2} + 2(t't'')^{1/2}[(1 + t'/4q^2)(1 + t''/4q^2)]^{1/2} \qquad (7\text{-}13)$$

The lower limit of t for nonvanishing $\rho(q^2,t)$ evidently occurs for $t' = t'' = t_0$, leading immediately to the boundary formula (7-2).

The pair of equations (7-12) and (7-8) for $\rho(q^2,t)$ and $f_2(q^2,t)$ completely determines the solution of the dynamical problem. To establish this circumstance, substitute (7-12) into (7-8) and interchange the order of integration:

$$f_2(t,q^2) = \rho_0(t) + \frac{1}{2\pi^2} \times \int_{t_0}^{(t')^{1/2} + (t'')^{1/2} < t^{1/2}} \int_{t_0} \frac{dt'\,dt''}{[t^2 + t'^2 + t''^2 - 2(tt' + tt'' + t't'')]^{1/2}} \times \int_{q^2_{\min}(t,t',t'')}^{\infty} \frac{dq'^2}{q'^2 - q^2} \frac{f_2^*(t',q'^2) f_2(t'',q'^2)}{[q'^2 - q^2_{\min}(t,t',t'')]^{1/2}} \qquad (7\text{-}14)$$

where

$$q^2_{\min}(t,t',t'') = \frac{tt't''}{t^2 + t'^2 + t''^2 - 2(tt' + tt'' + t't'')}$$

Now we see that under the integral a particular value of t' affects only those values of t such that $t > [(t')^{1/2} + (t_0)^{1/2}]^2$, and of course a corresponding remark can be made about t''. Such a situation then permits a stepwise

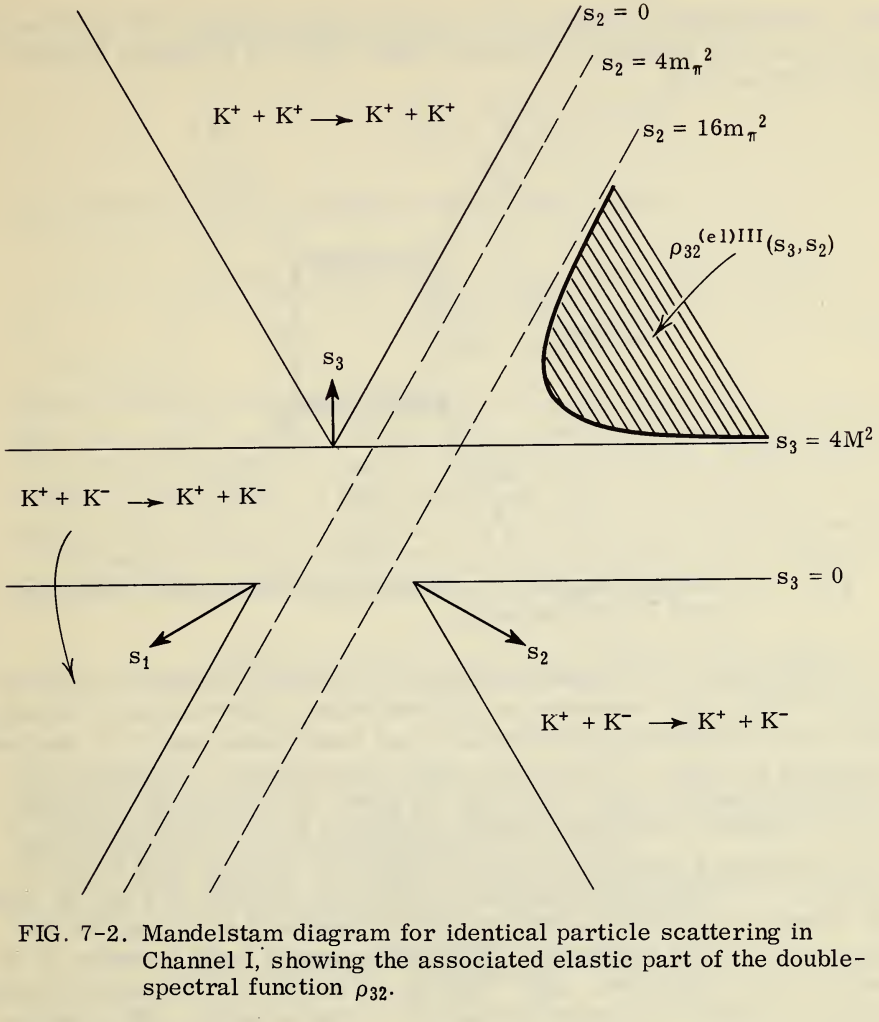

FIG. 7-2. Mandelstam diagram for identical particle scattering in Channel I, showing the associated elastic part of the double-spectral function ρ_{32}.

solution of (7-14), starting from the lowest values of t and working up. Explicitly, for $t_0 < t < 4t_0$,

$$f_2(t,q^2) = \rho_0(t) \tag{7-15}$$

so the range $4t_0 < t < 9t_0$ is completely given by substituting (7-15) into the right-hand side of (7-14). The new result then allows an extension to $9t_0 < t < 16t_0$, and so on. In general, n iterations cover the range $t_0 < t < n^2t_0$. The distance one has to go in order to obtain an adequate evaluation of the scattering amplitude through (7-9) depends on the asymptotic behavior of $f_2(t,q^2)$ for large t. This extremely important question will be the sub-

ject of discussion in Chapter 12, but we see that in principle the combination of unitarity and analyticity completely determines the dynamical solution once $\rho_0(t)$ is given.

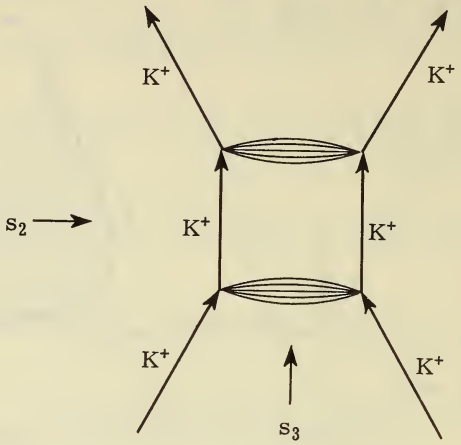

FIG. 7-3. Cutkosky diagram for the elastic part of the double-spectral function for Channel III.

Except for certain kinematical factors, the unitarity condition (7-3) will hold for the elastic absorptive part in relativistic scattering and, although the general Mandelstam representation has more terms than (7-1), the basic structure is similar. Thus we can expect again to have a dynamically determined situation once a "potential" is given. To identify the "potential" let us consider the example, already introduced in several connections, of the scattering of identical zero-spin particles of mass M (such as two K^+ mesons). The Mandelstam diagram for this case is shown in Fig. 7-2, where the domain of the elastic part of the double-spectral function ρ_{32} for the channel of interest is indicated. (Note the symmetry under exchange of s_2 and s_1.) The elastic part, $\rho_{32}{}^{(el)III}(s_3, s_2)$, is given by the Cutkosky diagram of Fig. 7-3, which is supposed to include *all* graphs having a $2K^+$ intermediate state in Channel III. The least-massive states that can occur in each of the exchange "blobs" are 2π states, so the asymptotes of the elastic double-spectral region are $4M^2$ and $16m_\pi{}^2$.

The Cutkosky rule may be used to evaluate $\rho_{32}{}^{(el)III}(s_3,s_2)$, but here we are interested in the potential analogy and so shall follow the original method of Mandelstam. We first define the elastic absorptive part for Channel III by†

†It will be shown later than one may associate the *entire* single-spectral function with the *elastic* absorptive part for the corresponding channel.

$$A_3^{(el)}(s_1,s_2,s_3) = \rho_3(s_3) + \frac{1}{\pi}\int ds_2' \frac{\rho_{32}^{(el)III}(s_3,s_2')}{s_2' - s_2} + \frac{1}{\pi}\int ds_1' \frac{\rho_{13}^{(el)III}(s_1',s_3)}{s_1' - s_1} \tag{7-16}$$

and write the unitarity condition corresponding to (7-3):

$$A_3^{(el)}(s_3, \cos\theta_3) = \frac{1}{4\pi}\frac{q_3}{(q_3^2 + M^2)^{1/2}}\int d\Omega'\, A^*(s_3, \cos\theta_3') \times A[s_3, \cos(\theta_3,\theta_3')] \tag{7-17}$$

Observe that this condition (by definition of $A_3^{(el)}$) holds for all s_3, and not just in the elastic region. The rest of the procedure is the same as for potential scattering; that is, we write $A(s_3, \cos\theta_3)$ in the form (3-7c) exhibiting the dependence on $\cos\theta_3$, substitute into (7-17), and carry out the angular integrations. One finds an analytic function of $\cos\theta_3$ with two cuts corresponding to the two integrals in (7-16) and identifies $\rho^{(el)}$ with the discontinuity. The result, after using the $2 \longleftrightarrow 1$ symmetry, is

$$\rho_{32}^{(el)III}(s_3,s_2) = \frac{1}{\pi}\frac{1}{q_3(q_3^2 + M^2)^{1/2}}\iint ds'ds'' \frac{A_2^*(s',s_3)A_2(s'',s_3)}{K^{1/2}(q_3^2; s_2,s',s'')} \tag{7-18}$$

where by $A_2(s',s_3)$ we mean $A_2(4M^2 - s' - s_3, s', s_3)$. This last formula is a close analogue of (7-12). As noted above, it could have been directly deduced from the Cutkosky prescription for Fig. 7-3.

Finally, the analogue of formula (7-8) is required in order to identify the "potential." If we introduce $\rho_{32}^{(in)III}(s_3,s_2)$ such that

$$\rho_{32}(s_3,s_2) = \rho_{32}^{(el)III}(s_3,s_2) + \rho_{32}^{(in)III}(s_3,s_2) \tag{7-19}$$

then one may write

$$A_2(s_2,s_3) = V_{III}^{II}(s_2,s_3) + \frac{1}{\pi}\int ds_3' \frac{\rho_{32}^{(el)III}(s_3',s_2)}{s_3' - s_3} \tag{7-20}$$

where

$$V_{III}^{II}(t,s_3) = \rho_2(t) + \frac{1}{\pi}\int ds_3' \frac{\rho_{32}^{(in)III}(s_3',t)}{s_3' - s_3} + \frac{1}{\pi}\int ds_1' \frac{\rho_{21}(t,s_1')}{s_1' - (4M^2 - s_3 - t)} \tag{7-21}$$

plays the role of the "potential" for Channel III generated by Channel II. That is, given V_{III}^{II} it is possible by the same iteration procedure as for nonrelativistic scattering to calculate the amplitude. In general, the "exchange potential" associated with Channel I is different from V_{III}^{II} but has a corresponding definition. In the particular example chosen here the direct and exchange "potentials" are the same.

Three differences between the relativistic and nonrelativistic problems should be noted. The first is the factor $(q_3^2 + M^2)^{1/2}$ in (7-18) which has been with us from the beginning. The second is the energy (s_3) dependence of V_{III}^{II} and the third, closely related to the second, is the imaginary part of V_{III}^{II} for s_3 above the threshold for inelastic processes:

$$\text{Im } V_{III}^{II}(t,s_3) = \rho_{32}^{(in)III}(s_3,t) \tag{7-22}$$

These differences are of physical importance, but if M^2 is much larger than m_π^2, as in nucleon-nucleon scattering, then one expects the approximation of replacing s_3 by $4M^2$ in both (7-18) and (7-21) to be reasonable for $q_3^2 \ll M^2$. In such an approximation an ordinary real energy-independent potential is achieved, as emphasized first by Charap and Fubini[23]:

$$V(r) = -\frac{1}{2M^2}\int dt\, \frac{\exp[-(t)^{1/2}r]}{r}\rho_0(t) \tag{7-23}$$

where†

$$\rho_0(t) = \rho_2(t) + \frac{1}{\pi}\int ds_3' \frac{\rho^{(in)III}(s_3',t)}{s_3' - 4M^2} + \frac{1}{\pi}\int ds_1' \frac{\rho_{21}(t,s_1')}{s_1' + t} \tag{7-24}$$

Solving the Schrödinger equation with such a potential would be equivalent to solving (7-18) and (7-20) by iteration. Unfortunately, for pion-pion and pion-nucleon scattering there is no hope of justifying an energy-independent potential and one must work with the integral equations.

Whether the problem is to solve a differential equation with an ordinary real potential or an integral equation with a complex energy-dependent potential, it is necessary to obtain the potential in the first place. We have a formula (7-21) in terms of certain single- and double-spectral functions, but how are these functions to be computed? Clearly, the next task is the formulation of a procedure to evaluate V_{III}^{II} and V_{III}^{I}.

†As discussed in reference 21 a correction may be made for the large s_3' region of the integral in Eq. (7-20).

8 EVALUATION OF THE LONG-RANGE AND MEDIUM-RANGE FORCES

In the previous section we defined a "potential" acting in Channel III, due to Channel II, as the absorptive part for Channel II minus an integral over the *elastic* double-spectral function $\rho^{(el)III}(s_3,s_2)$ for Channel III. Now in a number of cases we have seen that this elastic double-spectral function vanishes for $s_2 < s_2^{(min)'}$ where $s_2^{(min)'} > s_2^{(min)}$, the lowest mass squared in Channel II. Thus for the long-range components, corresponding to $s_2^{(min)} < s_2 < s_2^{(min)'}$, the "potential" is the entire absorptive part for Channel II extended to an unphysical region in s_3.

For N-N scattering in Channel III, $s_2^{(min)} = m_\pi^2$, corresponding to the single-pion intermediate state in N-$\bar{N}$ scattering, while $s_2^{(min)'} = 4m_\pi^2$. Furthermore the absorptive part here is a delta function in s_2, with no dependence on s_3, so there is no difficulty in the analytic continuation and the long-range potential is uniquely given for $m_\pi^2 < t < 4m_\pi^2$ by the pion-nucleon coupling constant. This circumstance has been widely exploited in analyses of the nuclear force problem.[20] For π-π scattering all three channels contain two pions and $s_2^{(min)} = 4m_\pi^2$, $s_2^{(min)'} = 16m_\pi^2$, so for $4m_\pi^2 < t < 16m_\pi^2$ the potential is the absorptive part for π-π scatterint itself and we have a "bootstrap" situation. That is, the long-range potential is not given a priori but is determined by the scattering that it produces! In a latter section we shall consider in some detail the coupled equations that describe this situation.

For π-N scattering the direct and exchange potentials are completely different since Channel II corresponds to $\pi + \pi \rightarrow N + \bar{N}$, with $s_2^{(min)} = 4m_\pi^2$, $s_2^{(min)'} = 16m_\pi^2$, while Channel I is again π-N scattering with $s_1^{(min)} = M^2$, $s_1^{(min)'} = (M + 2m_\pi)^2$, corresponding to the single-nucleon and single-nucleon plus two-pion intermediate states. Thus the long-range exchange potential is uniquely gkven by the pion-nucleon coupline constant but the long-

range direct potential requires the solution of a two-body dynamical problem. The exchange potential turns out to be the stronger of the two, as we shall see later, so some simple predictions can be made in terms of the coupling constant.

In order to calculate the potential in Channel III due to Channel II beyond the threshold $s_2^{(min)'}$ of $\rho_{32}^{(el)III}(s_3,s_2)$, one must exclude the portion of the absorptive part for Channel II that is associated with the Cutkosky graph of Fig. 7-3. Thus, in order to calculate the N-N potential for $4m_\pi^2 < t < 9m_\pi^2$ one needs the graphs of Fig. 8-1 *minus* the box graph (Fig. 4-2), which is part of Fig. 7-3 and therefore to be excluded from the left-hand graph of Fig. 8-1. A formula can be given by Cutkosky's rule for the graphs of Fig. 8-1 that is analogous to formula (7-18) for Fig. 7-3 but which involves instead the absorptive parts of π-N scattering.[5] A sufficient knowledge of pion-nucleon scattering, therefore, will allow a calculation of the N-N two-pion potential.

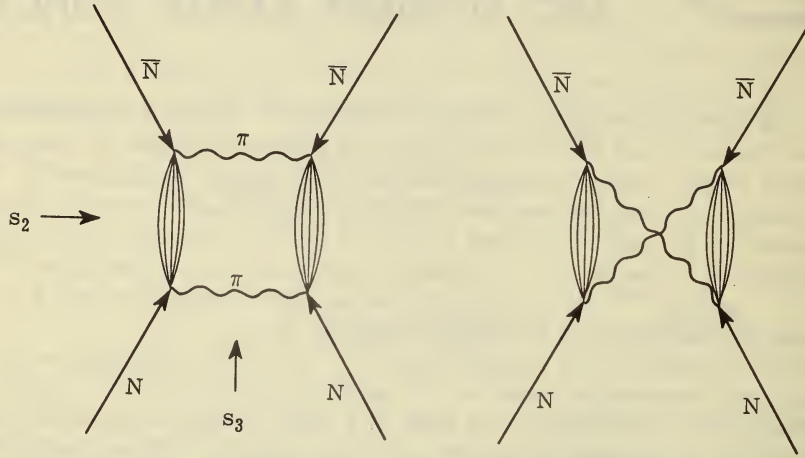

FIG. 8-1. Cutkosky graphs for 2π exchange in N-N scattering.

The above argument may be generalized, with the conclusion that any potential due to two-particle exchange can be expressed in terms of a bilinear product of absorptive parts of two-body transition amplitudes. It is uncertain, of course, how important the exchanges of higher multiplicity and shorter range are. In order to compute such effects we obviously will need to understand multibody systems, a development that may be slow in coming. There is reason to think, nevertheless, that a great deal will be learned from a thorough study of the long-range forces. Geometrically speaking, these forces should be dominant at both high and low energies and the framework described above is by no means restricted to the low-energy elastic domain. A theory has been constructed, appropriate to momentum transfers less than or on the order of $4m_\pi$ ($3m_\pi$ in some cases) but with no obvious restrictions on the energy.[21,24]

At the base of the theory is the π-π problem, which provides all its own long-range forces and an important part of many other two-body forces, such as the π-N force shown in Fig. 8-2. (We have already seen that the π-N amplitude in turn is a prerequisite for the N-N.)

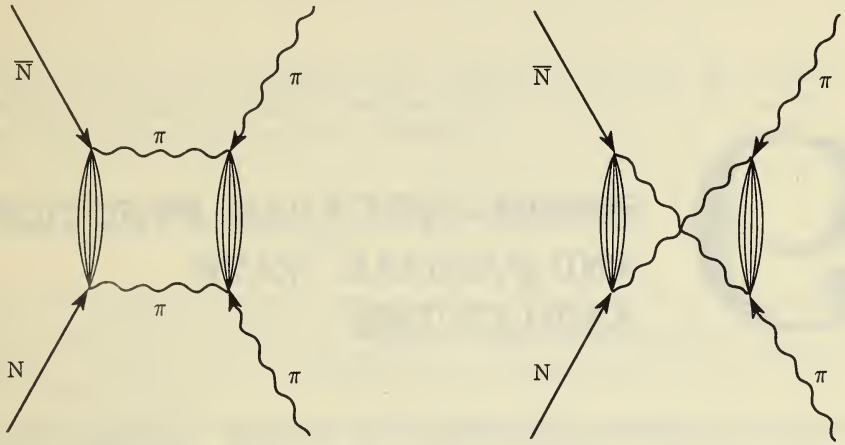

FIG. 8-2. Cutkosky graphs for 2π exchange in π-N scattering.

Thus, even though experimentally elusive, the π-π interaction cannot be avoided in the S-matrix approach, and we shall concentrate much of our attention on this problem in the following sections.

9 SINGLE-SPECTRAL FUNCTIONS AND PARTIAL-WAVE AMPLITUDES

It has been shown that a knowledge of the "potential" for Channel III is equivalent to a complete knowledge of the double-spectral function as well as the single-spectral functions for Channels II and I. Before the amplitude is determined, however, we also need the single-spectral function for Channel III. In nonrelativistic potential scattering this source of ambiguity is absent but it must be faced in the general problem.

The degree of arbitrariness remaining once the double-spectral function is known is severely restricted by unitarity. Later we shall consider the nature of this restriction in detail, but for the moment we simply state the result that has been established for all combinations of spin 0 and spin 1/2 particles: If the correct choice of invariant amplitudes has been made, the arbitrariness is at most of the form of the one-dimensional integrals in (3-4). Let us consider a case where all external spins are zero and examine the undetermined function

$$\frac{1}{\pi}\int ds_3' \frac{\rho_3(s_3')}{s_3' - s_3}$$

The first point to notice is the absence of dependence on s_2 and s_1 or, equivalently, of dependence on $\cos\theta_3$. Our uncertainty is therefore concentrated in the S wave of Channel III, once the potential for Channel III has been given. If we can calculate this S wave by some independent method the complete problem will have been solved.

In practice it may sometimes be desirable to calculate P or even D waves by the general technique to be described in the next section, even though these waves are determined by the double-spectral functions. Therefore, we shall now establish the location and strength of the singularities of an arbitrary partial-wave amplitude.

Let us consider equal-mass spin-zero particles and define a partial-wave amplitude for Channel III by the formula

$$A_l^{III}(q^2) = \frac{1}{2}\int_{-1}^{+1} d\cos\theta\ P_l(\cos\theta)\, A(q^2, \cos\theta) \tag{9-1}$$

where q^2 and $\cos\theta$ are related to the s variables by (2-8). In a certain region—including the physical interval, $q^2 > 0$, $-1 < \cos\theta < +1$—the full amplitude can be represented by the series

$$A(q^2, \cos\theta) = \sum_{l=0}^{\infty} (2l+1)\, A_l^{III}(q^2)\, P_l(\cos\theta) \tag{9-2}$$

but even outside this region we may define $A_l^{III}(q^2)$ by (9-1). We shall in fact be interested in extending $A_l(q^2)$ to the entire q^2 complex plane, where we shall find three sets of singularities, corresponding to the three channels of the problem.

Formula (6-1) is well suited to carrying out the projection (9-2). The first term of (6-1) contributes only to $l = 0$, and there it survives the projection unchanged. The second and third terms lead to integrals of the form

$$Q_{l\pm}(q^2, s') = \frac{1}{2}\int_{-1}^{+1} d\cos\theta\ P_l(\cos\theta)\, \frac{1}{s' + 2q^2(1 \pm \cos\theta)} \tag{9-3}$$

which are simply related to associated Legendre functions of the second kind. Certain important properties of these integrals may be seen by inspection:

1. For small q^2 they behave like $(q^2)^l$.
2. For large s' they behave like $(s')^{-l-1}$.
3. They are analytic functions of q^2 in the cut plane, where the cut should be chosen to run from $-s'/4$ to $-\infty$ if the function is to be real for $q^2 > 0$.

The behavior near $q^2 = 0$ is maintained after the integrations over ds_1' and ds_2' and is a well-known and general property of partial-wave amplitudes, related to the centrifugal barrier effect. The large s' behavior simply confirms our earlier remarks about the range of the interaction and the magnitude of s'; i.e., as l increases the large values of s_1' and s_2' become less and less important, since they correspond to short-range interactions. This is again a centrifugal barrier effect.

The property we are most concerned with is that of analyticity, and to get the full story here we have to look also at the q^2 (or s_3) dependence of the functions A_1 and A_2, which appear in the numerators of the integrands in formula (6-1). Referring to formula (3-6) for $A_l(s_1', 4M^2 - s_1' - s_3, s_3)$,

we see that the $s_3' - s_3$ denominator leads to a cut running from 0 to ∞ in q^2, while there seems to be a second cut associated with the denominator $s_2' - (4M^2 - s_3 - s_1')$. It may be verified, however, that this second apparent singularity is canceled by a corresponding part of the expression for $A_2(4M^2 - s_2' - s_3, s_2', s_3)$. After multiplying (9-3) by A_1 or A_2 and integrating over ds_1' or ds_2', we thus produce an analytic function of q^2 with two cuts, one running from 0 to ∞ along the positive real axis, to be called the "right-hand" or "physical" cut, and one running along the negative real axis from $-\infty$ to $-(1/4)s'_{min}$, where s'_{min} is the lowest square of a mass appearing in the spectra associated with Channels I and II. This latter will be called the "left-hand" or "unphysical" cut. In addition, if there exists a stable single-particle state with the quantum numbers of a particular partial wave of Channel III, there is a corresponding pole on the negative q^2 real axis.

In general, if the four particle masses are not all equal,[25] there are two distinct unphysical cuts, one for Channel I and one for Channel II, and these do not lie entirely on the real axis, but wander into the complex plane. The location of the cuts is always completely determined by kinematical considerations, however, and there are no new essential complications beyond the equal-mass case. We continue, therefore, to concentrate on that case.

Notice that our partial-wave amplitude is a real analytic function. That is, if we define $\nu = q^2$, we have

$$A_1^{III\,*}(\nu) = A_1^{III}(\nu^*) \tag{9-4}$$

Thus the discontinuity across a cut along the real axis is twice the imaginary part of the function at that point, while the function is real on the real axis in the gap between $-(1/4)s'_{min}$ and the origin. In the next section it will be shown that $A_1(\nu)$ is determined by the discontinuities across its cuts, so it is important to be able to calculate the imaginary part along the real axis. On the right-hand cut we are in the physical region and the required imaginary part is given by the unitarity condition. Comparing (9-2) with (2-2) and (2-3) we see that

$$\begin{aligned} A_1^{III}(\nu) &= [(\nu + m^2)/\nu]^{1/2}\, e^{i\delta_1} \sin\delta_1 \\ &= [(\nu + m^2)/\nu]^{1/2}\, [\eta_1 (e^{2i\alpha_1} - 1)/2i] \end{aligned} \tag{9-5}$$

if α_1 is the real part of the phase shift and $0 \le \eta_1 \le 1$. Equivalently,

$$\begin{aligned} \mathrm{Im}\, A_1^{III}(\nu) = {} & [\nu/(\nu + m^2)]^{1/2} |A_1^{III}(\nu)|^2 \\ & + [(\nu + m^2)/\nu]^{1/2} [(1 - \eta_1^2)/4] \end{aligned} \tag{9-6}$$

where the second term on the right-hand side represents inelastic scattering and is zero below the threshold ν_{in} for inelastic processes. This inelastic part of $\mathrm{Im}\, A_1^{III}$ may be calculated from the imaginary part of the

potentials:

$$\text{Im } A_1^{III\,(in)}(\nu) = [(\nu + m^2)/\nu]^{1/2} \frac{1 - \eta_1^2}{4}$$

$$= \frac{1}{\pi} \int dt' [\text{Im } V_I^{III}(t', \nu) Q_{1+}(\nu, t')$$

$$+ \text{Im } V_{II}^{III}(t', \nu) Q_{1-}(\nu, t')] \qquad (9\text{-}7)$$

On the left-hand cut we may most easily calculate the imaginary part from formula (6-1) before integrating over $\cos\theta$. Remembering that only the real parts of A_1 and A_2 contribute in this region, if the above-mentioned cancellation is anticipated, we find†

$$\text{Im } A_1^{III} = -1/2 \int_{-1}^{+1} d \cos\theta \; P_l(\cos\theta) \{\text{Re } A_1[-2\nu(1 + \cos\theta),$$

$$-2\nu(1 - \cos\theta), 4(\nu + m^2)] + \text{Re } A_2[-2\nu(1 + \cos\theta),$$

$$-2\gamma(1 - \cos\theta), 4(\nu + m^2)]\} \qquad \text{for } \nu < 0 \qquad (9\text{-}8)$$

Note that the possible presence of the first term in (6-1) for the case $l = 0$ does not affect either formula (9-6) or (9-8).

If we think of A_1 (or A_2) as made up of a sum of contributions from different kinds of intermediate states in Channel I (or II), as expressed by formula (6-3), then, according to (9-8), Im A_1^{III} is similarly composed of additive parts and these are nonzero along different portions of the negative real axis. It is evident, in fact, that a Channel I intermediate state of mass m_1 gives a nonzero contribution to Im A_1^{III}, according to (9-8), in the interval between $-\infty$ and $-(1/4)m_1^2$. Thus the least-massive intermediate states control Im A_1^{III} on the "nearby" portion of the left-hand cut. As we go farther to the left, more and more massive intermediate states come into the picture. Taking the nucleon-nucleon problem again as an example, we find the left-hand cut begins at $-1/4$, and up to -1 is completely determined by one-pion exchange. Between -1 and $-9/4$, the two-pion contribution must be added, between $-9/4$ and -4 the three-pion contribution, and so on. Each new threshold can be shown to be a branch point, with the associated cut running to the left.

The right-hand cut of the N-N amplitude of course begins at $\nu = 0$, where there is a branch point, and the next branch point does not occur until $\nu = (1/4)(2M + 1)^2 - M^2 \approx 7$, the threshold for single-pion production in nucleon-nucleon collisions. In the $J = 1$ even-parity amplitude there is a pole corresponding to the deuteron and falling in the gap between cuts at $\nu = (1/4)M_D^2 - M^2 \approx -(1/10)$. We shall see that this pole need not be postulated in advance, but is a necessary consequence of the left and right cuts.

†We define the partial-wave amplitude along both cuts as the limit as the cut is approached from *above*.

10 DETERMINATION OF PARTIAL-WAVE AMPLITUDES FROM UNPHYSICAL AND INELASTIC DISCONTINUITIES

Given the direct and exchange "potentials" for a particular channel (III) we have seen above in formula (9-8) how to calculate the imaginary part of A_1^{III} along the unphysical left-hand and cut: let us call the imaginary part on this cut $f_1(\nu)$. We have also seen in formula (9-8) how to calculate the quantity $\eta_1(\nu)$ which measures the amount of inelastic scattering. Our problem now is to determine $A_1^{III}(\nu)$ in terms of the given quantities $f_1(\nu)$ and $\eta_1(\nu)$. We shall follow a procedure proposed by Chew and Mandelstam[26] for the case $\eta_1 = 1$ and generalized by Froissart,[27] which was an outgrowth of a technique used in the original work of Chew and Low.[6] An alternative approach which has the same physical content has been developed by Omnes.[28]

Let us begin with the case $\eta_1 = 1$ and define the real analytic Omnes function[28]

$$D_1(\nu) = \exp\left[-\frac{\nu - \nu_0}{\pi}\int_0^\infty d\nu' \frac{\alpha_1(\nu')}{(\nu' - \nu)(\nu' - \nu_0)}\right] \qquad (10\text{-}1)$$

normalized to unity at the arbitrary point $\nu_0 \leq 0$ on the negative real axis. This function is real on the negative real axis but for $\nu > 0$ has the phase $-\alpha_1$, where we establish the convention $\alpha_1(0) = 0$. By construction $D_1(\nu)$ is analytic in the entire complex ν plane, cut along the positive real axis, and has no zeros. The tacit assumption is also made that a limit of $\alpha_1(\nu)$ as $\nu \to \infty$ exists, so that the asymptotic behavior of $D_1(\nu)$ is†

† The importance of this consideration was pointed out in a private conversation by M. Froissart.

$$D_1(\nu) \xrightarrow[\nu \to \infty]{} \exp\left[\frac{\alpha_1(\infty)}{\pi} \ln \nu + \text{const.}\right]$$
$$\longrightarrow \text{const.} \times \nu^{\alpha_1(\infty)/\pi} \tag{10-2}$$

Next we define a function $N_1(\nu)$ by

$$N_1(\nu) = A_1^{III}(\nu)\, D_1(\nu)$$

or

$$A_1^{III}(\nu) = N_1(\nu)/D_1(\nu) \tag{10-3}$$

a definition that leads to the nomenclature of calling N_1 the "numerator" function and D_1 the "denominator" function. We may immediately infer from (10-3) that $N_1(\nu)$ is a real analytic function with the same left-hand cut as $A_1^{III}(\nu)$ but with no right-hand cut, since for $\nu > 0$ the phase of D_1^{-1} is exactly the same as that of A_1^{III}. Thus

$$\begin{aligned} \text{Im } N_1(\nu) &= f_1(\nu)\, D_1(\nu) \qquad \text{for } \nu < \nu_L \\ &= 0 \qquad\qquad\qquad\quad \text{for } \nu_L < \nu \end{aligned} \tag{10-4}$$

if ν_L is the end of the left-hand cut. Observe now from Eq. (9-6) that with $\eta_1 = 1$

$$\text{Im}[A_1^{III}(\nu)]^{-1} = -[\nu/(\nu + m^2)]^{1/2} \qquad \nu > 0 \tag{10-5}$$

It then follows that

$$\begin{aligned} \text{Im } D_1(\nu) &= -[\nu/(\nu + m^2)]^{1/2}\, N_1(\nu) \qquad \text{for } \nu > 0 \\ &= 0 \qquad\qquad\qquad\qquad\quad \text{for } \nu < 0 \end{aligned} \tag{10-6}$$

Now we are in a position to write down Cauchy integral formulas—or dispersion relations—for N_1 and D_1, if we can establish the asymptotic behavior. From formula (9-5) it is clear that for large positive ν the absolute value of $A_1^{III}(\nu)$ is bounded by unity. Since we are representing A_1^{III} by a quotient, however, it is not clear what restrictions are to be placed on N_1 and D_1 separately. More explicitly, according to (10-2) we require that

$$\nu^{-\alpha_1(\infty)/\pi} N_1(\nu) \tag{10-7}$$

be bounded as $\nu \to \infty$ but we have no a priori restriction on the values of $\alpha_1(\infty)$. The particular equations we write down for N_1 and D_1 therefore will depend on the value of $\alpha_1(\infty)$. This ambiguity was first emphasized by Castillejo, Dalitz, and Dyson,[29] and we shall see that it is associated with the possibility of unstable "independent" particles having the quantum numbers of Channel III.

The simplest assumption to make is that $\alpha_1(\infty) < \pi$, so that

$$\lim_{\nu \to \infty} (1/\nu)\, D_1(\nu) = 0$$

and

$$\lim_{\nu \to \infty} (1/\nu)\, N_1(\nu) = 0 \tag{10-8}$$

We also assume that there are no poles in $A_1^{III}(\nu)$, i.e., no stable particles with angular momentum 1 and the quantum numbers of Channel III. In such a case we may write the following Cauchy formulas, since the restrictions (10-8) must hold for all directions in the complex plane with the possible exception of the negative real axis:

$$N_1(\nu) = A_1^{III}(\nu_0) + \frac{\nu - \nu_0}{\pi} \int_{-\infty}^{\nu_L} d\nu' \frac{f_1(\nu')\, D_1(\nu')}{(\nu' - \nu)(\nu' - \nu_0)} \tag{10-9}$$

$$D_1(\nu) = 1 - \frac{\nu - \nu_0}{\pi} \int_0^{\infty} d\nu' (\nu'/\nu' + m^2)^{1/2} \frac{N_1(\nu')}{(\nu' - \nu_0)(\nu' - \nu)} \tag{10-10}$$

The solution of these linear integral equations, if it exists at all, is believed to be unique.[26] The question of existence depends on the behavior of the discontinuity along the left cut, $f_1(\nu)$, as $\nu \to -\infty$. In particular, it is easy to show by a Pomeranchuk type of argument† that if Im $A_1^{III}(\nu)$ approaches a limit along the left cut then it must approach the same limit along the right cut. On the right, however, the imaginary part is positive definite and bounded by unity. Thus if a limit of $f_1(\nu)$ exists as $\nu \to -\infty$, then $0 \leq f_1(-\infty) \leq 1$. Chew and Mandelstam[26,30] found that such an asymptotic condition leads to a unique solution of (10-9) and (10-10). However, if $f_1(\nu)$ oscillates indefinitely as $\nu \to -\infty$, no simple statement can be made. At present the asymptotic behavior of $f_1(\nu)$ is not well understood except in the nonrelativistic limit of ordinary potential scattering,[22] where (10-9) and (10-10) have been shown to yield the same solution as would be obtained by solving the Schrödinger equation. [In this case $f_1(-\infty) = 0$.]

The appearance of the parameter ν_0 in our equations does not correspond to any arbitrariness in the amplitude A_1^{III}, since it merely specifies the point at which D_1 is normalized to unity. Changing ν_0 changes the normalization of D_1, but since N_1 will change by the same factor, the quotient is left unaltered. The subtraction constant $A_1^{III}(\nu_0)$, however, requires some

†See the Appendix. The application of such an argument to partial-wave amplitudes was made by Chew and Mandelstam, reference 30.

discussion. The centrifugal barrier effect was shown in Chapter 9 to give the general requirement that $A_1^{III}(\nu) \sim \nu^1$ near $\nu = 0$. Thus by choosing $\nu_0 = 0$ we may eliminate the subtraction constant for all partial waves except the S, but for the latter, one free real parameter appears inescapable in the relativistic problem. In the following chapter this free parameter will be related to the notion of a "coupling constant." [For nonrelativistic scattering no such parameter appears since, when the factor $(\nu + m^2)^{1/2}$ is replaced by m, the partial-wave scattering amplitude must vanish at infinity.]

Froissart[27] has shown how to generalize the foregoing method to the case when $\eta_1 \neq 1$. He begins by defining a real analytic function

$$R_1(\nu) = \exp\left[-i\frac{\nu^{1/2}}{\pi}\int_0^\infty d\nu' \frac{\log \eta_1(\nu')}{(\nu')^{1/2}(\nu' - \nu)}\right] \tag{10-11}$$

with the right-hand cut only and with a modulus along this cut equal to η_1. The phase of $R_1(\nu)$ along the cut is

$$\varepsilon_1(\nu) = -\frac{\nu^{1/2}}{\pi} P\int_0^\infty d\nu' \frac{\log \eta_1(\nu')}{(\nu')^{1/2}(\nu' - \nu)} \tag{10-12}$$

Froissart then observes that if we define $A_1^{III'}(\nu)$ by

$$1 + 2i\left(\frac{\nu}{\nu + m^2}\right)^{1/2} A_1^{III'}(\nu) = R_1^{-1}(\nu)\left[1 + 2i\left(\frac{\nu}{\nu + m^2}\right)^{1/2} A_1^{III}(\nu)\right] \tag{10-13}$$

then along the physical cut,

$$[\nu/(\nu + m^2)]^{1/2} A_1^{III'}(\nu) = \{\exp[i(2\alpha_1 + \varepsilon_1)] - 1\}/2i \tag{10-14}$$

so $A_1^{III'}$ obeys a condition of the form (10-5) and the previous method may be applied to the modified amplitude. The imaginary part of $A_1^{III'}$ along the left cut is found by an easy calculation to be

$$f_1'(\nu) = R_1^{-1}(\nu)\left[f_1(\nu) - \frac{1}{2}\left(\frac{\nu + m^2}{\nu}\right)^{1/2}\right] + \frac{1}{2}\left(\frac{\nu + m^2}{\nu}\right)^{1/2} \tag{10-15}$$

If a partial-wave amplitude exists with the prescribed discontinuities and the prescribed asymptotic behavior then we shall find it by the method outlined. It is possible, however, that no such function exists and yet the Eqs. (10-9) and (10-10) still can be solved. Such a situation occurs when the forces are attractive and sufficiently strong to produce one or more bound states with angular momentum 1. The function $A_1^{III}(\nu)$ will then have

corresponding poles in the gap $-m^2 < \nu < 0$ which are not included in the specification of singularities based on a knowledge of $f_l(\nu)$ and $\eta_l(\nu)$. In such a situation, even if we do not know in advance whether the poles should be present, they will appear automatically in the solution of our integral equation through zeros of the denominator function.† In other words, we shall find a function that does *not* satisfy the specification of singularities that led to our equations but which is nevertheless the physically correct solution. This is the sense in which we said earlier that the deuteron pole is a consequence of other singularities. If f_l and η_l are correctly given, both the position and the residue of a bound-state pole follow from the solution of Eqs. (10-9) and (10-10).

To illustrate these considerations let us consider nonrelativistic scattering with $\eta_l = 1$. No subtraction is required here for the numerator function, so we may replace (10-9) by the simpler relation

$$N_l(\nu) = \frac{1}{\pi}\int_{-\infty}^{\nu_L} d\nu' \frac{f_l(\nu')\, D_l(\nu')}{\nu' - \nu} \tag{10-16}$$

We study the case $l = 0$ and make a very crude approximation in which $f_l(\nu)$ is represented by a delta function. In other words, we approximate the left-hand cut (a line charge) by a pole (a point charge). This approximation is reasonable for a region along the positive real axis whose extent is short compared with the "average" distance to the important left-hand singularities. Figure 10-1 shows some of the distances for the case of n-p scattering. We see that if the 1π exchange force is not too strong, the replacement of the left-hand cut by a pole may be reasonable for kinetic energies (lab) of 0 to 10 Mev.

Suppose we locate the interaction pole at $\nu = -\nu_i$ and normalize D_0 to unity at this point (i.e., choose $\nu_0 = -\nu_i$). Then if we introduce a parameter Γ, to characterize the strength of the interaction, by writing

$$f_0(\nu) = -\pi\Gamma\delta(\nu + \nu_i) \tag{10-17}$$

we have, first, from (10-16), and, second, from (10-10),

$$N_0(\nu) = \Gamma/(\nu_i + \nu) \tag{10-18}$$

$$D_0(\nu) = 1 - \frac{\Gamma}{\pi}(\nu + \nu_i)\int_0^{\infty} d\nu' \left(\frac{\nu'}{\nu' + m^2}\right)^{1/2} \frac{1}{(\nu' + \nu_i)^2(\nu' - \nu)} \tag{10-19}$$

† The function D_l given by Eq. (10-10), when there is a bound state at $\nu = \nu_B$, differs from the Omnes function (10-1) by a factor $(\nu_B - \nu)/(\nu_B - \nu_0)$.

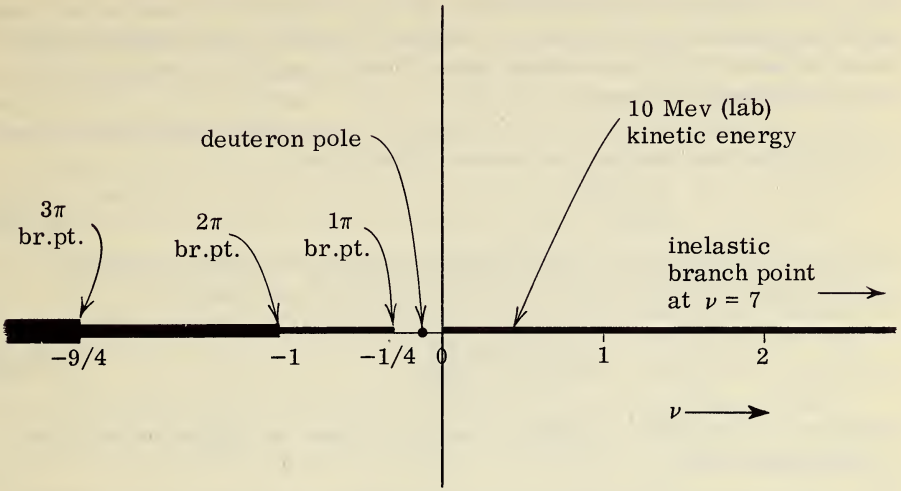

FIG. 10-1. "Nearby" singularities of a partial-wave n-p amplitude.

so in this simple case we do not even have to solve an integral equation. The integral in (10-19) can easily be performed and the nonrelativistic result (i.e., for ν and ν_i both small compared with m^2—an excellent approximation for n-p scattering where $m^2 = 44$) is

$$D_0(\nu) = 1 - \frac{\Gamma}{m} \frac{\nu + \nu_i}{2\nu_i^{1/2}[\nu_i^{1/2} + (-\nu^{1/2})]^2} \tag{10-20}$$

In the physical region, $\nu > 0$, we then find

$$\mathrm{Re}\frac{D_0(\nu)}{N_0(\nu)} = \frac{\nu^{1/2}}{m} \cot\delta_0 = \left(\frac{\nu_i}{\Gamma} - \frac{\nu_i^{1/2}}{2m}\right) + \nu\left(\frac{1}{\Gamma} + \frac{1}{2m\nu_i^{1/2}}\right) \tag{10-21}$$

Comparing this with the standard nonrelativistic effective-range formula,

$$q \cot\delta_0 = (1/a) + (1/2)rq^2$$

we see

$$1/a = (m/\Gamma)\nu_i - (1/2)\nu_i^{1/2} \tag{10-22a}$$

$$(1/2)r = (m/\Gamma) + (1/2\nu_i^{1/2}) = (1/\nu_i^{1/2}) + (1/a\nu_i) \tag{10-22b}$$

Let us study the dependence of our result on the input parameters Γ and ν_i. First, if ν_i is held fixed and Γ is small, we see that the scattering length a is proportional to Γ and has the same sign, exactly what we expect if Γ determines the magnitude and sign of the interaction. Evidently

positive Γ means an attractive force. If Γ is negative (repulsive), increasing its magnitude does not make a indefinitely large; the scattering length never exceeds $2/\nu_i^{1/2}$ in absolute value. This is exactly the behavior of a repulsive potential of range $\sim 2/\nu_i^{1/2}$.

If Γ is positive (attractive), increasing its magnitude makes the scattering length increase and in fact become infinite for

$$\Gamma = 2m\nu_i^{1/2} \tag{10-23}$$

This is the condition for a bound state to appear; for larger values of Γ one may easily calculate from (10-20) that $D_0(\nu)$ has a zero at

$$-\nu = \alpha^2 = \nu_i \left(\frac{\Gamma - 2m\nu_i^{1/2}}{\Gamma + 2m\nu_i^{1/2}} \right)^2 \tag{10-24}$$

Thus we can calculate the binding energy—if there is a bound state—from a knowledge of f_0.

Problems: 1. Show that, for our single-interaction pole example, when there is a bound state one may write

$$\nu^{1/2} \cot \delta_0 = -\alpha + (\nu + \alpha^2)[(m/\Gamma_B) + (1/2\alpha)]$$

where Γ_B is the residue of the bound-state pole and $-\alpha^2$ its position.

2. By comparison with the nonrelativistic formula for the effective range in terms of an integral over the square of the bound-state configuration-space wave function, identify the residue Γ_B with the square of the asymptotic normalization coefficient for the bound-state function.

Finally we remark that if we are near the condition for a bound state, so that the scattering length is large, then according to (10-22) the effective range is approximately $2/\nu_i^{1/2}$. This is a second confirmation that the inverse distance to an unphysical singularity corresponds roughly to the interaction range.

In the actual case of n-p scattering the effective-range formula is extremely accurate in the interval 0 to 10 Mev, and the empirically observed values of the scattering length and effective range for both singlet and triplet states have been shown by Noyes and Wong[31] to imply a value of ν_i in the above formula approximately equal to unity. In other words the "average" position of the left-hand discontinuity in the n-p amplitude occurs near the beginning of the 2π contribution. This is an understandable circumstance if both 1π and 2π forces are important.

A two-pole approximation of the left cut is of course better than a one-pole approximation and the problem may again be reduced to an algebraic rather than an integral equation, as it can for any finite number of poles. It

is not clear how far such an approach can profitably be pushed and we shall not further pursue it here. We have discussed the pole approximation not because it is an essential part of the S-matrix theory of strong interactions but merely because it serves to illustrate certain properties of Eqs. (10-9) and (10-10).

A final remark is perhaps in order before we consider how to modify these equations when the assumption (10-8) is abandoned. The remark is that in practice we shall never have available the complete "potentials" from which to calculate η_l and f_l but, at best, we can calculate the long- and medium-range parts. We shall always be forced either to neglect or to represent in some phenomenological way certain short-range forces. Experience has demonstrated and it is geometrically to be expected that the shorter the range the less important is the force, so we have good reason to hope that the "peripheral" approach is in principle convergent. Nevertheless we shall in many situations be trying to solve Eqs. (10-9) and (10-10) with inaccurate input functions $f_l(\eta)$ and $\eta_l(\nu)$. In such a circumstance one must always check the solution that emerges to confirm that it does not have spurious singularities; our derivation guarantees a physically sensible solution only when the input is correct and consistent with assumption (10-8).

Now we turn to the most obscure and confusing aspect of the S-matrix approach to particle dynamics: the number of independent parameters permitted. In this chapter we have touched on two related points, the value of the phase shift at ∞ and the S-wave subtraction constant. Earlier we discussed the residues of poles in terms of "coupling constants." Let us now survey the whole situation to see if a general understanding can be achieved.

11 ARBITRARY PARAMETERS IN THE S MATRIX

We are now in a position to attempt to characterize the parameters left undetermined by the combined requirements of unitarity and analyticity. This attempt cannot be entirely successful because we have not gone outside the two-body subspace of the S matrix, and even within this subspace certain questions remain obscure. However the considerations already brought out allow a surprisingly small degree of arbitrariness, as originally emphasized by Mandelstam.[10] One central consideration, mentioned above but not explained, is the extent to which the amplitude is determined once the double-spectral function is known. Let us begin here by reviewing the work of Froissart.[32]

Froissart starts by proving that (for zero-spin particles) the amplitude $A(s_1,s_2,s_3)$ in the physical region for Channel III is bounded by

$$A \leq Cs_3 \log^2 s_3 \qquad \text{at forward and backward angles} \qquad (11\text{-}1)$$

and

$$A \leq Cs^{3/4} \log^{3/2} s_3 \qquad \text{at any other fixed angle} \qquad (11\text{-}2)$$

C being a constant. Corresponding statements can, of course, be made for Channels I and II. The proof is based on the unitarity limit for partial-wave amplitudes plus the observation that for the Mandelstam representation to make sense the single- and double-spectral functions cannot increase faster than some power. We shall not give the proof here but only repeat Froissart's explanation of its physical content for forward elastic scattering in terms of the notion of potential. In configuration space we have for large r a (direct) potential

$$V(r) \sim g(s_3)(e^{-r/R}/r) \qquad (11\text{-}3)$$

where $R \sim (s_2^{\min})^{1/2}$ and where, according to (7-21) and (7-23), $g(s_3)$ increases at most as a power of s_3. If $g e^{-r/R}$ is small compared to unity there will be practically no scattering; if large there will be strong scattering. Thus the cross section will essentially be determined by πa^2, where a is the value of r for which $g e^{-r/R} \sim 1$; i.e., $a \sim R \log g$. The conclusion from this qualitative argument, then, is that

$$\sigma_3^{\text{tot}} \sim \pi R^2 \log^2 g$$

or

$$\sigma_3^{\text{tot}} \leq C \log^2 s_3 \tag{11-4}$$

The optical theorem tells us that

$$\text{Im } A(s_3, \theta_3 = 0) = (q_3 s_3^{1/2}/8\pi)\, \sigma_3^{\text{tot}}$$

so we conclude from (11-4) that $\text{Im } A \leq C s_3 \log^2 s_3$ in the forward direction, and the real part turns out to be similarly bounded. (For an individual partial wave the real and imaginary parts are separately bounded by unity.)

A clue as to why the amplitude has the more-restrictive bound (11-2) at a fixed angle not forward or backward is given by the observation that

$$\sigma_3^{\text{tot}} = \sigma_3^{\text{el}} + \sigma_3^{\text{in}}$$

while

$$\sigma_3^{\text{el}} = (4/s_3) \int d\Omega_3 \, |A(s_3, \theta_3)|^2$$

$$= (16\pi/s_3) \langle |A|^2 \rangle_{\text{av}} \tag{11-5}$$

Thus if $A \sim s_3$ at all angles, then $\sigma_3^{\text{el}} \sim s_3$, in violation of (11-4). Such reasoning, however leads only to the conclusion that

$$\langle |A|^2 \rangle^{1/2} \leq C s_3^{1/2} \ln s_3 \tag{11-6}$$

and puts no bound on the behavior at any particular angle as does (11-2). A precise physical meaning for the latter condition is lacking, although it emerges from Froissart's analysis in a straightforward way.

Experimentally, total cross sections appear to approach constant limits at high energy, so the maximum behavior allowed by elementary considerations of analyticity and unitarity is closely approached for elastic scattering in the forward direction. (Whether logarithmic factors occur is not yet clear experimentally.) For backward directions, however, and generally for reactions that are not strictly elastic it seems probable that the actual asymptotic behavior in either forward or backward directions has the more-

restrictive bound characteristic of other angles. The point is that, physically, the forward direction has a special significance only for coherent elastic scattering where the phenomenon of diffraction occurs. Further analysis of the S matrix can be expected to give a mathematical basis for such a statement; for the moment we shall treat it as a supplementary physical postulate.†

Given the restrictions (11-1) and (11-2) Froissart then proceeds to analyze the arbitrariness left in the amplitude once the double-spectral functions are known and concludes that at most two subtractions in each variable may be regarded as arbitrary. Again the proof will not be given but only a plausibility argument. Suppose that one wished to include in the Mandelstam representation a term of the form

$$s_3^2 \int ds_2' \frac{\rho_2^{(2)}(s_2')}{s_2' - s_2} \tag{11-7}$$

corresponding to having made three subtractions in s_3. In the forward direction for Channel III the variable s_2 is fixed (at the value zero) so such a term, taken alone, violates the restriction (11-1) and there must be a correlation with the remaining double-dispersion integrals to produce a cancellation at large s_3. Using both requirements (11-1) and (11-2) Froissart shows that the necessary correlation determines the subtraction term (11-7) uniquely; in this sense the subtraction should not have been made in the first place. A paradox appears to arise when the asymptotic behavior of the double-spectral function is such as to *require* a subtraction if the integral in which it appears is to be everywhere defined by the usual standards. The meaning of Froissart's result is that a definition can always be given to such integrals, without subtraction, by analytic continuation, presumably because the integrand oscillates even when its modulus increases beyond bound.

The elimination by the above argument of all subtraction terms of an order higher than that of (11-7) is obvious, but the case of one lower order,

$$s_3 \int ds_2' \frac{\rho_2^{(1)}(s_2')}{s_2' - s_2} \tag{11-8}$$

is marginal. Here one does not violate condition (11-1) but in cases such as π-π, π-N, and π-K one violates the more-severe condition mentioned above on charge exchange or backward scattering. (This circumstance will be explored in our detailed discussion of the π-π problem.) In fact one can find reasons to exclude (11-8) in all cases studied so far, and there are

†Such a postulate is closely related to a principle suggested by Pomeranchuk[33] that any particular target has the same high-energy cross section for all projectile particles within a given isotopic multiplet.

grounds to suspect that a general argument will be forthcoming. Thus we are left only with the arbitrariness associated with single subtraction terms of the type written in Eq. (3-4), plus an over-all subtraction constant for zero spins, which is equivalent to an arbitrariness in the S partial-wave amplitude.

In the previous section the analyticity properties of partial-wave amplitudes were analyzed and a procedure set up for determining these functions in terms of f_l and η_l, quantities derivable from the generalized potentials. The procedure, however, depended on assuming (1) that $\alpha_l(\infty) < \pi$ and (2) that there are no stable elementary particles with the quantum numbers in question. (It was found that a bound-state pole would develop automatically from the equations derived.) Let us now concentrate our attention on the case $l = 0$ and relax these assumptions in order to find the most-general solution.

As a matter of *definition* we call the solution of Eqs. (10-9) and (10-10) "pure potential scattering," since in the nonrelativistic limit it has been shown that this is identical with the solution of the ordinary Schrödinger equation in which the corresponding nonrelativistic potential has been employed.[22] As a first and simplest variation let us consider the possibility of a single stable spin-zero "independent" particle (i.e., not a bound state of the particular two-body system under consideration) of mass m_p with the quantum numbers of Channel III,† but at the same time we maintain assumption (1) above. To obtain the corresponding solution for $A_0^{III}(\nu)$ we merely add a pole term,

$$\Gamma_p/(\nu_p - \nu) \tag{11-9}$$

to the right-hand side of Eq. (10-9), where $\nu_p = m_p^2/4 - m^2$. If we normalize D_0 to unity at ν_p (i.e., set $\nu_0 = \nu_p$), then Γ_p is the residue of the pole in the over-all amplitude and therefore is given by the square of the coupling constant linking the particle of mass m_p to the two particles of Channel III.

The new solution depends on the two additional parameters, m_p and Γ_p, and is distinct from a solution of the "pure potential" type, even one that happens to yield a bound state at ν_p. [In the latter case, for example, one would find $\alpha(\infty) < \alpha(0)$.] In principle, then, we may distinguish between independent particles and bound states by asking whether the associated scattering amplitude does or does not require a pole term to be added in Eq. (10-9). There is no principle known, however, to restrict the number of independent particles, and each that occurs will bring two arbitrary constants.

†There are no known examples of this type for spin-zero scattering, but in π-N scattering the nucleon may play an analogous role for the $J = \frac{1}{2}$ P state.

Next let us change assumption (1) to

$$\pi \leq \alpha_0(\infty) < 2\pi \tag{11-10}$$

In this case, according to (10-2) and (10-7) we shall need one additional subtraction in both N_0 and D_0 and, correspondingly, two new parameters enter the problem. One can show that N_0 and D_0 each acquire a zero as a result of the subtraction, the position of the zero being controlled by the value of the subtraction constant. If the zero in D_0 happens to occur on the real axis in the interval $-m^2 < \nu < 0$, then one has achieved exactly the same solution as obtained by adding (11-9) to the numerator function. [The D functions obtained by the two methods differ by a factor $\propto(\nu_p - \nu)$, but the N functions differ by the same factor.] We again are describing a situation with a stable independent particle and merely have replaced m_p and Γ_p by the two subtraction constants. It is, however, possible to choose the subtraction constants so that the zero of D appears on the unphysical sheet of the complex plane.[28] The associated pole in the amplitude is usually described as an "unstable independent particle," since it is a natural extension of a stable-particle pole.

Changing the condition (11-10) to $n\pi \leq \alpha < (n + 1)\pi$ evidently leads to n pairs of arbitrary constants and n poles. To summarize, the ambiguity in our S-wave amplitude is associated with the possibility of zero-spin independent particles, either stable or unstable, having the Channel III quantum numbers. Without some new principle the number of such particles can only be determined on an experimental basis. With each independent particle is associated two real constants—corresponding to the mass and to the rate of decay (partial width) into the two particles of Channel III. (When the particle is stable we have the reduced width—which is essentially the same as the square of the coupling constant.)

For problems such as π-N, where half odd-integer values of angular momentum occur, the $J = 1/2$ states play a role analogous to that described here for the S state. All higher partial waves are uniquely determined by the double-spectral functions. In the sense of the definitions proposed above, therefore, the combined requirements of unitarity and analyticity appear to preclude the existence of strongly interacting elementary particles of spin greater than 1/2. A correspondence will immediately be noticed with conventional renormalization theory where it has not been possible to accommodate higher-spin particles without very special restrictions on the form of their interactions. Such restrictions, if translated into the language of the S matrix presumably mean that the existence of a higher-spin particle with the Channel III quantum numbers manifests itself in Channels I and II in such a way that the appropriate Channel III partial-wave amplitude, even though calculated by analytic continuation from the double-spectral function, nevertheless has a pole in the correct place. The detailed mechanism for such a phenomenon, if it occurs in nature and is distinct from a bound state,

remains a mystery. To date, no promising candidate for "elementary" status has appeared among strongly interacting particles of spin greater than 1/2.

To conclude this chapter something should be said about the subtraction constant $A_1^{III}(\nu_0)$ needed in Eq. (10-9) even when there are no independent particles to be considered. A deep physical meaning for this constant is lacking, although it certainly corresponds to the renormalizable quadrilinear (four scalar fields) interaction of conventional field theory. No such constant appears in the spin 0-spin 1/2 interaction because the partial-wave amplitude there is forced to vanish at infinity by the unitarity condition.[34] It has also been shown for the spin 1/2-spin 1/2 problem that no free constant occurs.[16] In both cases there is a correspondence with conventional field theory where the corresponding quadrilinear interactions are not renormalizable. For spin 0-spin 0 scattering, $A_1^{III}(\nu_0)$ is often called a "coupling constant," although the terminology is misleading since the point ν_0 is arbitrary. Whatever the terminology, however, the combined requirements of unitarity and analyticity do permit one such free parameter for each pair of spin-0 particles.

The masses and partial widths of independent particles evidently also appear as arbitrary parameters in conventional field theory (the partial widths are the squares of renormalized trilinear coupling constants). Thus we see that the S-matrix approach at this stage has exactly the same degree of arbitrariness as conventional renormalizable field theory, no more and no less.

12 PION-PION SCATTERING: GENERAL FORMULATION

Let us turn now to a specific consideration of π-π scattering, the simplest strong-interaction problem. The diagram in question is that of Fig. 12-1, where we use the isotopic vector index running from 1 to 3 to label the pion-charge degree of freedom.† By assuming charge independence the complete amplitude may be written as the sum of three terms,

$$A(s_1,s_2,s_3)\delta_{\alpha\beta}\delta_{\gamma\delta} + B(s_1,s_2,s_3)\delta_{\alpha\gamma}\delta_{\beta\delta} + C(s_1,s_2,s_3)\delta_{\alpha\delta}\delta_{\beta\gamma} \tag{12-1}$$

while crossing symmetry gives the relations

$$\left.\begin{matrix} A \longleftrightarrow A \\ B \longleftrightarrow C \end{matrix}\right\} \quad s_1 \longleftrightarrow s_2 \qquad s_3 \longleftrightarrow s_3$$

$$\left.\begin{matrix} A \longleftrightarrow B \\ C \longleftrightarrow C \end{matrix}\right\} \quad s_2 \longleftrightarrow s_3 \qquad s_1 \longleftrightarrow s_1 \tag{12-2}$$

$$\left.\begin{matrix} A \longleftrightarrow C \\ B \longleftrightarrow B \end{matrix}\right\} \quad s_1 \longleftrightarrow s_3 \qquad s_2 \longleftrightarrow s_2$$

The connection between A, B, C and the amplitudes for well-defined isotopic spin in Channel III turns out to be

†Our notation generally will follow as closely as possible the paper of Chew and Mandelstam.[26]

$$A^{III,0} = 3A + B + C$$
$$A^{III,1} = B - C \qquad (12\text{-}3)$$
$$A^{III,2} = B + C$$

Comparing these with the first of the crossing relations (12-2) we see that since $\cos\theta_3 \rightarrow -\cos\theta_3$ as $s_1 \leftrightarrow s_2$, the amplitudes for isotopic spin 0 and 2 are even functions of $\cos\theta_3$, while that for isotopic spin 1 is an odd function, in accordance with Bose statistics. Henceforth the channel index III is to be understood except when the contrary is explicitly stated. The index T will be used to denote isotopic spin.

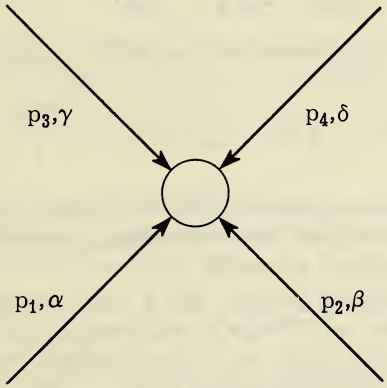

FIG. 12-1. Diagram for π-π scattering.

The elastic (3,2) double-spectral function for isotopic spin T will be written $\rho_T{}^{(el)}(s_3,s_2)$ and because of crossing symmetry the corresponding (3,1) elastic double-spectral function differs at most by a sign. Specifically, the two are related by the factor $(-1),^T$ so we have

$$A_3{}^{(el)T}(s_3,s_2) = \rho_3{}^T(s_3) + \frac{1}{\pi}\int ds_2' \frac{\rho_T{}^{(el)}(s_3,s_2')}{s_2' - s_2}$$
$$+ \frac{(-1)^T}{\pi}\int ds_1' \frac{\rho_T{}^{(el)}(s_3,s_1')}{s_1' - s_1} \qquad (12\text{-}4)$$

realizing that by crossing symmetry $\rho_3{}^1(s_3)$ must vanish. The complete absorptive part for Channel II with isotopic spin T in Channel III shall be denoted by $\tilde{A}_2{}^T(s_2,s_3)$ while that for Channel I differs only by the factor $(-1)^T$. Thus formula (7-18) may be used,

$$\rho_T{}^{(el)}(s,t) = \frac{1}{\pi q_s (q_s{}^2 + 1)^{1/2}} \iint dt'\, dt'' \frac{\tilde{A}_2{}^{T*}(t',s)\,\tilde{A}_2{}^T(t'',s)}{K^{1/2}(q_s{}^2;\, t,t',t'')} \qquad (12\text{-}5)$$

with the integration limit (7-13) to be understood. Another formal result needed is the "crossing matrix" that expresses $\tilde{A}_2{}^T$ in terms of $A_3{}^T$ at a corresponding point. This may be obtained by combining Eqs. (12-3) and (12-2); one finds

$$\tilde{A}_2{}^T(s,t) = \sum_{T'} \beta_{TT'} A_3{}^T(s,t) \tag{12-6}$$

with

$$\beta_{TT'} = \begin{pmatrix} 1/3 & 1 & 5/3 \\ 1/3 & 1/2 & -5/6 \\ 1/3 & -1/2 & 1/6 \end{pmatrix} \tag{12-7}$$

Finally we define the generalized potential $V_T(t,s)$ by the equation

$$\tilde{A}_2{}^T(t,s) = V_T(t,s) + \frac{1}{\pi}\int ds' \frac{\rho_T{}^{(el)}(s',t)}{s'-s} \tag{12-8}$$

following the prescription given by formula (7-20). All the preceding remarks have been formal and, correspondingly, exact. Our first real task is to evaluate the generalized potential V_T.

An analysis of Cutkosky graphs, taking into account the absence of any 3π vertex, reveals that $\rho_T{}^{(el)}(s_3,s_2)$ requires at least four-pion intermediate states in Channel II. Thus, in an approximation that neglects forces due to transfer of four or more pions, we have

$$V_T(t,s) \approx \tilde{A}_2{}^{(el)T}(t,s) = \sum_{T'} \beta_{TT'} A_3{}^{(el)T'}(t,s) \tag{12-9}$$

This formula is exact for $t < 16$, but it may be necessary to add supplementary terms to represent short-range forces, just as in the nucleon-nucleon problem it has been necessary to insert a phenomenological "hard core." We see in any event that the longer-range forces are determined by formula (12-4), so that a "bootstrap" mechanism is operating.

In order to evaluate formula (12-4), even supposing that $\rho_T{}^{(el)}$ has been given by some sort of iteration procedure, it is necessary to obtain $\rho_3{}^0$ and $\rho_3{}^2$, the single-spectral functions which have been seen above to be related to the S-wave amplitudes. Specifically, if the S-wave parts are separated from the integrals in Eq. (12-4) we may write

$$A_3{}^{(el)T=0,2}(s,t) = \mathrm{Im}\, A_0{}^{(el)T=0,2}(s) + \frac{1}{\pi}\int dt' \rho_T{}^{(el)}(s,t') \times \left[\frac{1}{t'-t} + \frac{1}{t'-4+s+t} - \frac{1}{2q_s{}^2}\ln\left(1+\frac{4q_s{}^2}{t'}\right)\right] \tag{12-4'}$$

where, according to Eq. (9-6),

$$\text{Im } A_0^{(el)T}(s) = [(s-4)/s]^{1/2} |A_0^T(s)|^2 \qquad (12\text{-}10)$$

In the preceding lecture a procedure for calculating the S wave was developed in terms of the discontinuity on the left cut $f_0^T(s)$ and the absorption parameter $\eta_0^T(s)$. At this point, therefore, formulas for these quantities are required. From Eq. (9-8) we have

$$f_0^T(s) = \frac{1}{2q_s^2} \int_4^{-4q_s^2} dt \, \text{Re } \tilde{A}_2^T(t,s) \qquad s < 0 \qquad (12\text{-}11)$$

and from Eq. (9-7)

$$\left(\frac{s}{s-4}\right)^{1/2} \frac{1-{\eta_0^T}^2}{4} = \frac{1}{\pi} \int dt \frac{1}{2q_s^2} \ln\left(1 + \frac{4q_s^2}{t}\right) \text{Im } V_T(t,s)$$

$$s > 16 \qquad (12\text{-}12)$$

To this point all formulas in this last paragraph are exact. In fact the *only* approximation so far has been in formula (12-9) for the generalized potential, which implies that in formula (12-12)

$$\text{Im } V_T(t,s) \approx \sum_{T'} \beta_{TT'} \rho_{T'}^{(el)}(t,s) \qquad (12\text{-}9')$$

Before turning to the question of how one might try to solve all these coupled equations, we must have clearly in mind the minimum number of independent parameters—even assuming that the approximation (12-9) is adequate. In the preceding section it was explained that even if one assumes that there are no elementary particles with the quantum numbers of two pions, an S-wave subtraction constant still remains arbitrary. One might expect two such constants here because there are two different S states (T = 0,2), but the crossing relations (12-2) provide a link. In particular, at the symmetry point, $s_1 = s_2 = s_3 = 4/3$, it is evident that A = B = C. One conventionally defines the pion-pion "coupling constant" as the negative of the value of the amplitude at this point:

$$\lambda \equiv -A(4/3, 4/3, 4/3) = -B(4/3, 4/3, 4/3)$$
$$= -C(4/3, 4/3, 4/3) \qquad (12\text{-}13)$$

Since for all channels the symmetry point corresponds to $\cos\theta = 0$, $q^2 = -2/3$, we have, from Eqs. (12-3),

$$A^{T=0}(\nu = -2/3, \cos\theta = 0) = -5\lambda$$
$$A^{T=1}(\nu = -2/3, \cos\theta = 0) = 0 \qquad (12\text{-}14)$$
$$A^{T=2}(\nu = -2/3, \cos\theta = 0) = -2\lambda$$

It is expected that at the symmetry point the amplitudes will be dominated by their S-wave components, so to a good approximation,

$$A_0^{T=0}(\nu = -2/3) \approx -5\lambda$$
$$A_0^{T=2}(\nu = -2/3) \approx -2\lambda \qquad (12\text{-}15)$$

and there is no difficulty in correcting for the higher-wave contributions. Thus if we choose $\nu_0 = -2/3$ in Eqs. (10-9) and (10-10) we see that a single real constant λ should determine π-π scattering if there are no "hidden" independent particles.

To recapitulate and at the same time describe the only systematic approach yet devised for the solution of our equations let us imagine that the free parameter λ is "turned on" adiabatically, starting from a value zero. For $\lambda = 0$ there exists the consistent if trivial solution of the equations in which all functions vanish everywhere. Now suppose λ is given a small value; the first-order effect seen in Eq. (10-9) is to produce constant S-wave amplitudes equal to -5λ and -2λ, respectively, for $T = 0$ and $T = 2$. According to Eqs. (12-10), (12-4′), and (12-9), the real part of the potential then assumes a value proportional to λ^2.

Given the weak real potential V_T one solves by iteration the dynamical equations (12-5) and (12-8) to obtain $\tilde{A}_2^T$, whose real part grows $\sim\lambda^2$ but whose imaginary part, $\rho_T^{(e)}$, grows only $\sim\lambda^4$. Now we may use formula (12-11) to compute f_0^T, which is $\sim\lambda^2$, noting that at this stage η_0^T is still equal to unity, according to formula (12-12). Finally, going back to solve Eqs. (10-9) and (10-10) we see that the corrections to our first approximation will nowhere be greater than order λ^2, so by sufficient repetition of the cycle for sufficiently small λ convergence should be obtained. The complete amplitude may then be calculated from the formulas

$$A^{T=0,2}(s,t) = A_0^{T=0,2}(s) + \frac{1}{\pi}\int dt'\, \tilde{A}_2^T(t',s)\left[\frac{1}{t'-t} + \frac{1}{t'-4+s+t} - \frac{1}{2q_s^2}\ln\left(1+\frac{4q_s^2}{t'}\right)\right]$$

$$A^{T=1}(s,t) = \frac{1}{\pi}\int dt'\, \tilde{A}_2^{T=1}(t',s)\left(\frac{1}{t'-t} - \frac{1}{t'-4+s+t}\right) \qquad (12\text{-}16)$$

This first solution for a small λ may be used as a trial function to start the iteration procedure for a slightly larger value—and so on. If the increments in λ are made sufficiently small it should be possible to continue finding solutions at least until the asymptotic behavior of the amplitude changes in a fundamental way from the way it behaves with λ small. Let us consider now this question of asymptotic behavior, which at the moment is one of the crucial problems of strong-interaction physics.

13 ASYMPTOTIC BEHAVIOR OF AMPLITUDES: THE PRINCIPLE OF MAXIMUM STRENGTH FOR STRONG INTERACTIONS

To ease gently into the tricky subject of asymptotic behavior let us list the behavior when λ is small of the various functions introduced in the preceding section. We shall concern ourselves only with powers of s and t and systematically ignore possible logarithmic factors. It has been noted that the S-wave amplitudes are constants, so it follows that to order λ^2

$$\begin{aligned} \text{Re } V_T(t,s) &\propto \text{const.} \qquad t \to \infty, \text{ s finite} \\ &\propto \text{const.} \qquad s \to \infty, \text{ t finite} \end{aligned} \tag{13-1}$$

Evidently, from formula (12-8), the same statement may be made for Re $\tilde{A}_2^T(t,s)$. Then from formulas (12-5) and (7-11) it may be deduced that

$$\rho_T^{(el)}(s,t) \propto \text{const.} \qquad t \to \infty, \text{ s finite} \tag{13-2}$$

In fact one may, from an examination of the function K and the upper limits of the integral in (12-5), conclude that if any power is assumed for the asymptotic behavior of $\tilde{A}_2^T(t,s)$ as $t \to \infty$, then the same power will control the corresponding asymptotic behavior of $\rho_T^{(el)}(s,t)$.

In contrast, if $\tilde{A}^T(t,s) \propto$ const. as $s \to \infty$, then Eq. (12-5) tells us that

$$\rho_T^{(el)}(s,t) \propto 1/s \qquad s \to \infty, \text{ t finite} \tag{13-3}$$

This difference in the asymptotic behavior of $\rho_T^{(el)}(s,t)$ with respect to its two arguments is of great importance. The behavior (13-2) means, for example, that even for λ small we must make the subtraction in (12-4′) in

order that the integral containing $\rho_T^{(el)}$ be defined. At the same time the behavior (13-3) guarantees a meaning for Eq. (12-8) as well as formula (12-12), without any subtractions.

We cannot rest easy at this point, however, because experiment tells us emphatically that in nature the actual asymptotic behavior is not that described above. Remembering the optical theorem,

$$A_3^T(s,0) = (q_s s^{1/2}/8\pi)\sigma_{tot}^T(s) \tag{13-4}$$

and the fact that experimental total cross sections seem to approach constants, we are forced to the conclusion from (12-6) and (12-7) that the $T = 0$ "crossed" absorptive part has a behavior

$$\tilde{A}_2^0(t,0) \propto t \qquad t \to \infty \tag{13-5}$$

The asymptotic behavior of the corresponding $T = 1$ and $T = 2$ functions may be less than t because of a cancellation between positive and negative elements of the matrix $\beta_{TT'}$. In fact, if we accept the conjecture of Chapter 11 that charge-exchange cross sections approach zero at infinity, then σ_{tot}^0, σ_{tot}^1, and σ_{tot}^2 all approach the same limit and one sees from (12-7) that a cancellation actually will occur for $\tilde{A}_2^1$ and $\tilde{A}_2^2$. However, nothing can reduce the behavior (13-5) for $\tilde{A}_2^0$ if the total cross sections approach constants, so the analysis made on the basis of small λ is definitely inadequate.

A clue as to how the "crossed" absorptive part $\tilde{A}_2^T(t,s)$ can develop an asymptotic behavior $\propto t^\alpha$, $\alpha > 0$, for large t and finite s, even when $V_T(t,s)$ is well behaved in this limit, has been given by Regge in an analysis of nonrelativistic potential scattering.[35] It will be recalled that in Chapter 7 we showed that if the s dependence of the generalized potential $V_T(t,s)$ in formula (12-8) is neglected and the factor $(q_s^2 + 1)^{1/2}$ replaced by unity in Eq. (12-5), these two equations are identical with the dynamical equations for nonrelativistic potential scattering. Since we are interested in the large t behavior of $\tilde{A}_2^T(t,s)$ for s near the threshold, it is plausible that Regge's analysis, based on the Schrödinger equation, is relevant to the general problem. What Regge showed was that $f_2(t,q^2)$, the analogue of $\tilde{A}_2^T(t,s)$, behaves like $t^{\alpha(q^2)}$ as $t \to \infty$, where the real part of α increases with increasing strength of the finite t components of an attractive potential, even though the potential goes to zero for large t. The power α varies with q^2 and is real for $q^2 < 0$ but complex for q^2 positive. For negative q^2, if $\alpha(q_B^2) = \alpha_B$ is a positive integer, then there is a bound state for $l = \alpha_B$, with a binding energy corresponding to q_B^2. On the other hand, if the real part of α is equal to a positive integer α_R for some positive value $q^2 = q_R^2$, then we have a resonance at this energy for $l = \alpha_R$. (The width of the resonance is related to the imaginary part of α.) Increasing the strength of an attractive

potential evidently makes possible bound states and resonances in states of higher and higher angular momentum. Thus the asymptotic power α can be made arbitrarily large by making the long-range attractive forces sufficiently strong.

It is plausible then that in the π-π problem

$$A_2{}^T(t,s) \propto t^{\alpha_T(s)} \tag{13-6}$$

where α_T depends on the strength of the generalized potential V_T. It has been shown by Froissart[36] that such behavior is consistent with the dynamical equations (12-5) and (12-8) provided α_T has an imaginary part for $s > 4$. Gribov has emphasized, on the basis of these same equations, that if α_T is real, the asymptotic behavior must have an additional factor $(\ln t)^{-\beta}$, where $\beta > 1$.[37] If α_T is complex, any power of $\ln t$ appears possible. We shall continue the discussion, however, without considering logarithmic factors. There is no experimental evidence for them and they are absent in potential scattering, the source of most of our intuition about dynamics.

Combining the statements (13-5) and (13-6) we see that according to experiment,

$$\alpha_0(s = 0) = 1 \tag{13-7}$$

We may also remember from Chapter 11 the limitation (11-1) of Froissart, which states that

$$\alpha_0(s = 0) \leq 1 \tag{13-8}$$

This circumstance, together with the assumption that the magnitude of α_T reflects the strength of the potential V_T, has led Chew and Frautschi[11] to propose the following general principle for determining the magnitude of strong interactions: *Forces between strongly interacting particles have the maximum strength consistent with the unitarity and analyticity of the S matrix.* More specifically, it is suggested that if one insists in general scattering problems on the condition corresponding to Eq. (13-7), then dimensionless parameters such as λ, up to now regarded as arbitrary, will in fact be determined.

A philosophical basis for the Chew-Frautschi principle can be expressed through the ancient notion of "lack of sufficient reason." The Landau-Cutkosky rules for the location and strength of singularities of the S matrix in no way distinguish "elementary particles" from bound states or dynamical resonances. Furthermore, in all cases experimentally explored to date, where one can distinguish in the sense of Chapter 11 between the alternatives of a certain particle being "complex" or "elementary," the decision has

gone in favor of the former. There appears, then, no good "reason" to invoke the notion of elementary particle for strong interactions. Now without the elementary-particle concept to focus attention on particular singularities of the S matrix, the question immediately arises: Where does one begin the dynamical calculation? The answer is that it is only a matter of convenience; one may begin anywhere, taking an arbitrary singularity as a starting point and attempting to reach as much of the S matrix from this point as computational ability allows. Generally speaking, of course, it is convenient to begin with the lowest-mass singularities. A second question is: What determines the strength of the "starting" singularities? There is nothing special about these singularities, so it is plausible to appeal once more to the notion of "lack of sufficient reason." Since the singularity strength is bounded by unitarity, it seems natural to postulate that strong interactions "saturate" the unitarity condition. There seems no *reason* for any other particular strength to occur.

14 PION-PION DYNAMICS

Returning to the π-π problem we recall that the one free parameter λ characterizes the magnitude of the low-energy S-wave amplitudes and thus through (12-9) the strength of the longest-range forces. Referring to formula (12-7) we see that for $T = 0$ these long-range forces are attractive (they give a positive $V_{T=0}$), so one is in a position to make α_0 increase by increasing the magnitude of λ. Estimates made by Chew, Mandelstam, and Noyes[38] show that λ must be negative if appreciable forces are to result, but it was also found that the S-wave part of $A_3^{(el)}$ in (12-9) by itself cannot give sufficient attractive force to satisfy (13-7). It appears, however, that the P-wave ($T = 1$) part of $A_3^{(el)}$—which also gives an attractive although somewhat shorter-range contribution to $V_{T=0}$—may be sufficiently strong to do the job† if there is a resonance in this ($J = 1$, $T = 1$) state.[39]

A possible mechanism to solve the π-π problem by the adiabatic approach might then be the following: The parameter λ is increased in the negative sense giving weak long-range attractive forces in all three states, according to (12-7). At some point the P amplitude may grow sufficiently to give a contribution to the forces comparable to that from the S states. If this happens, then henceforth there will be a "bootstrap" situation as the force due to P exchange increases to the magnitude necessary for (13-7). That is, the chief attractive part of $V_{T=1}$ that makes the P wave large is

†The question as to whether a particular contribution to the force acting in S states is attractive or repulsive is confused in the relativistic zero-spin problem by the necessity for an S-wave subtraction. In the discussion here we define "attraction" and "repulsion" with respect to the generalized potential V_T, since α_T is sensitive only to this quantity. However, if one is interested in the effect on the S wave in particular, caution is required in using these notions.

due to the $T = 1$, $l = 1$ part of $A_3^{(el)}$ in formula (12-9). Note that this force, owing to exchange of P-wave pairs, is more attractive in $V_{T=0}$ than in $V_{T=1}$ and actually repulsive for $V_{T=2}$. It is reasonable then to expect that $\alpha_0(0) > \alpha_1(0) > \alpha_2(0)$, the kind of situation anticipated above in the discussion following (13-5). That is, if we can reach the desired goal of (13-7) we may expect to find at the same time

$$\alpha_1(0),\ \alpha_2(0) < 1 \tag{14-1}$$

corresponding to

$$\sigma_{tot}{}^{T=0}(\infty) = \sigma_{tot}{}^{T=1}(\infty) = \sigma_{tot}{}^{T=2}(\infty)$$

The reader may be concerned that if a P resonance is to be achieved we shall violate (14-1). It must be recalled, however, that $\alpha_T(s)$ varies with s, so although a P resonance at $s = s_R$ implies that

$$\text{Re } \alpha_1(s_R) = 1 \tag{14-2}$$

there is no conflict with (14-1) if $\alpha(s)$ is an increasing function; such is in fact the case for nonrelativistic scattering by an attractive potential. A second source of possible confusion is the observation that condition (13-7) seems to imply a zero-energy bound P state with $T = 0$. Because of the Bose statistics, however, there *is* no $T = 0$ P state.

There exists the possibility, therefore, of a theory of π-π scattering that depends on the pion mass and nothing else. However, we have neglected short-range forces involving kaons and nucleons and until the dynamical equations actually are solved one does not know how to assess the role of such forces. The chief obstacle so far to the numerical solution of the equations is associated with the imaginary part of the asymptotic power α_T. An oscillatory behavior is implied that is essential, when Re $\alpha_T(s) \geq 1$, to give a meaning to various integrals [for example the integrals over ds_1' and ds_2' in Eq. (12-4)], but which is difficult to handle numerically. Techniques to cope with asymptotic oscillations are being intensively studied at the present time.

All our discussion of the π-π problem has tacitly assumed that we are dealing with "pure potential scattering" in the sense of Chapter 11. If unstable "elementary particles" exist with the quantum numbers of two pions, then corresponding poles must be inserted on the unphysical sheets for the appropriate partial waves. Each such particle brings with it two arbitrary real parameters—corresponding to its mass and decay width. It goes without saying that the existence of such particles would undermine the principle of "maximum strength." Up to the present time fragmentary experimental information about the π-π system suggests two virtual states, one with $J = 1$, $T = 1$ and a mass $\sim 4.5 m_\pi$ and one with $J = 0$, $T = 0$ and a mass

$\sim 2m_\pi$.[40] Both these states, however, appear to be dynamical resonances—i.e., to arise from zeros of the denominator function defined by Eq. (10-10). The former resonance we have discussed above as playing a central role in the dynamics; the role of the latter in generating forces is less important because of the small statistical weight associated with its quantum numbers, but there seems no reason for surprise that such a state should exist.[41] A clear conclusion as to whether these two states are elementary or complex must, however, await a solution of the dynamical equations.

15 THE VERTEX FUNCTION OF ONE VARIABLE: PION ELECTROMAGNETIC STRUCTURE AS AN EXAMPLE

If we are completely to exclude weak and electromagnetic interactions from our considerations there would be no need to study the so-called vertex functions—which are associated with three-particle diagrams. These have entered our strictly strong-interaction approach only when all three particles are on the mass shell—where one is dealing with a single number, not a function. If we want to take advantage, however, of the established validity of lowest-order perturbation theory with respect to the fine-structure constant and the Fermi weak-interaction constant, in discussing certain experiments involving photons or leptons as well as strong interactions, then we shall find it necessary to analyze three-particle vertices in cases where the mass of a lepton pair or a photon is considered a complex variable.

Suppose, for example, that we want to describe the deviations from Coulomb scattering of an electron by a charged pion, deviations due to the "structure" of the pion associated with its strong interactions. That is to say, a real pion can emit virtual strongly interacting systems of appropriate quantum numbers that produce a charge distribution in its neighborhood. By electron elastic scattering of sufficiently large momentum transfer we should be able to probe the "structure" of this charge distribution and learn something about the strong interactions that produce it, if we assume that the electromagnetic aspects of the situation are completely understood. The latter statement can be given a concrete meaning in terms of Fig. 15-1, which represents all diagrams in which a single photon is exchanged between the electron and the pion; multiple photon exchanges should be weaker in order of magnitude by a factor $e^2 = 1/137$. The single-photon part of the amplitude, ignoring electron and photon spin,† can be written

†Since the pion has zero spin, Fig. 15-1 involves only one invariant function of t even when the electron and photon are correctly described.

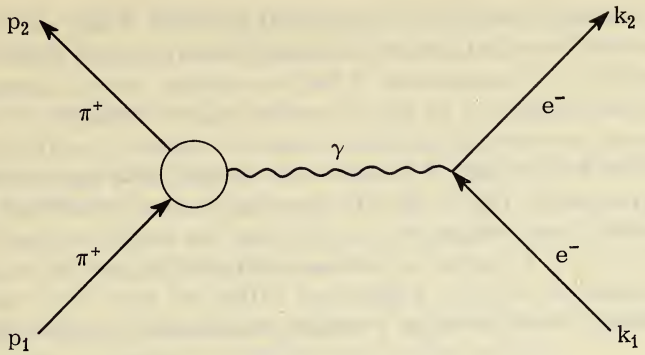

FIG. 15-1. Diagram for electron-pion elastic scattering via a single virtual photon.

$$F_{\pi^+\pi^-}{}^{\gamma}(t)\, S^{\gamma}(t)\, F_{e^+e^-}{}^{\gamma}(t) \tag{15-1}$$

just as for Fig. 3-2, where S^{γ} is the photon propagator and $F_{\pi^+\pi^-}{}^{\gamma}$ and $F_{e^+e^-}{}^{\gamma}$ are vertex functions, all three depending on

$$t = (p_1 - p_2)^2 = (k_1 - k_2)^2 = -2q^2(1 - \cos\theta) \tag{15-2}$$

if as usual q and $\cos\theta$ are the barycentric-system variables. What is meant when we say we "understand" the purely electromagnetic aspects of the problem is the assumption that, to an accuracy of order e^2, we have

$$S^{\gamma}(t) = 1/t \tag{15-3}$$

and

$$F_{e^+e^-}{}^{\gamma}(t) = e \tag{15-4}$$

There *are* of course known modifications of the photon propagator and the electron-photon vertex that play an important role in low-t electrodynamical experiments of high accuracy; however, these modifications, being of order e^2, are much smaller than expected effects from the pion-photon vertex.

It is not certain that (15-3) and (15-4) are correct for the large values of t in which we shall be interested, since these formulas have been tested only for relatively small t. A failure for large t is what is commonly referred to as "a breakdown of quantum electrodynamics at short distances," and a search for such a breakdown is an important objective in current electron accelerator research. No evidence has yet been developed, how-

ever, that we cannot trust (15-3) and (15-4) to order e^2 for all t; therefore, we shall assume here that a measurement of electron-pion elastic scattering is essentially a measurement of the pion-photon vertex function.

The physical range of t in elastic scattering, according to (15-2), is $t < 0$. However, one can, in addition, measure electron-positron annihilation to produce a $\pi^+\pi^-$ pair, a process that according to the substitution law is also determined by (15-1). In this case the physical range is $t > 4$. We can in principle, then, measure $F_{\pi^+\pi^-}(t)$ over the whole real axis except for the gap, $0 < t < 4$; so let us now see what predictions can be made about this vertex function, which is sometimes called the pion "form factor," since classically it would be the Fourier transform of the pion charge density.

The principles to be used can be applied to many other vertex problems, involving nucleons rather than pions and weak rather than electromagnetic "probing" interactions. Actually the nucleon-photon vertex is the one for which most data are available, but we shall see in the following chapter that to understand nucleon structure a prior understanding of pion structure is required. In addition there are the usual complications due to nucleon spin that tend to obscure the essential ideas involved.

It can be shown by a study of the Landau rules[4] that a vertex function of one variable, except for a possible subtraction, satisfies the representation

$$F(t) = \frac{1}{\pi}\int dt' \frac{g(t')}{t' - t} \tag{15-5}$$

where the real spectral function g(t) is nonvanishing† only for t equal to the square of the mass of a possible intermediate physical state having the quantum numbers of the single particle whose mass is the variable, as well as of the pair of particles on the other side of the vertex. The single external particle itself, however, is to be excluded from the spectrum. In our example we need to enumerate strongly interacting states having the quantum numbers of a photon as well as a $\pi^+\pi^-$ pair. Even though an external photon is involved, we need not worry about photons in intermediate states, because these give contributions to our spectral function g(t) which are small (of order e^2).

Using charge independence and charge-conjugation symmetry, as well as other well-known conservation laws, we conclude that for the pion form factor states must have $T = 1$, $T_3 = 0$, $J = 1$, be odd under both space reflection and charge conjugation, and of course have zero strangeness and zero baryon number. The lightest such state is the P-wave pion pair, then we

†As in the case of the Mandelstam representation, there may be anomalous thresholds for certain mass ratios. (See, for example, Karplus, Sommerfield, and Wichmann.[42]) We shall not consider such cases here.

jump to 4,6,8,..., pions, $K\bar{K}$ pairs, etc. It follows that the lower limit of integration in (15-5) is at $t' = 4$ and that $F_\pi(t)$ is real for $t < 4$. For $t > 4$, we have

$$\text{Im } F_\pi(t) = g_\pi(t) \tag{15-6}$$

so if we remember that $F_\pi(t)$ is proportional to and has the same phase as the matrix element

$$\langle \pi^+(p_2),\ \pi^-(-p_1) | T | \gamma(p_2 - p_1) \rangle$$

where the fictitious initial state consists of a photon with mass $(t)^{1/2}$, and also remember the unitarity condition (4-2), then we see

$$g_\pi(t) = (\text{real const.}) \times \sum_m \langle m | T | \pi^+(p_2),\ \pi^-(-p_1) \rangle^* \times \langle m | T | \gamma(p_2 - p_1) \rangle \tag{15-7}$$

For $4 < t < 16$, only the 2π intermediate state contributes, and we have

$$g_\pi(t) = (\text{real const.}) \times A_1^{1*}(t) F_\pi(t) \qquad \text{for } 4 < t < 16 \tag{15-8}$$

where A_1^1 is the $T = 1$, $l = 1$, partial-wave π-π amplitude introduced earlier [but considered here as a function of $t = 4(\nu + 1)$, which is a more convenient variable]. Now $g_\pi(t)$ is supposed to be real, so according to (15-8) the vertex function $F_\pi(t)$ must have the same phase as the P-wave amplitude along the real axis for $4 < t < 16$; and to the extent that we do not expect inelastic π-π scattering to compete seriously for $t \lesssim 40$,† the simple phase condition should be approximately correct over the wider interval.

More precisely, consider the function

$$G(t) = F_\pi(t) D_1^1(t) / e D_1^1(0) \tag{15-9}$$

Remembering that $D_1^1(t)$ has the phase $-\alpha_1^1$ we see that $G(t)$ is analytic in the entire complex t plane except for a cut running along the real axis from 16 to ∞. Furthermore $G(0) = 1$, since gauge invariance guarantees $F_\pi(0) = e$, and although the asymptotic behavior of $F_\pi(t)$ is not yet understood it seems likely that this function is bounded. Thus if π-π scattering is of the "potential" type, so that $t^{-1} D_1^1(t)$ approaches zero as t goes to

† Experience with π-N scattering suggests that, until each of the produced pions can have a momentum in the barycentric system of order m_π, phase-space factors will hold the inelastic cross section to a small fraction of the elastic.

infinity, we may write down a dispersion relation for G(t) with a single subtraction:

$$G(t) = 1 + \frac{t}{\pi}\int_{16}^{\infty} dt' \frac{\text{Im } G(t')}{t'(t' - t)} \tag{15-10}$$

The conjecture that inelastic π-π scattering is small for $t \lesssim 40$ means that Im G(t) is expected to be small for $16 < t \lesssim 40$, so we may approximate (15-10) by

$$G(t) \approx 1 + t/\zeta_{in} \tag{15-11}$$

where ζ_{in} is the real constant whose sign is unknown but which in magnitude is expected to be greater than, or about, 40. In their calculation of the pion factor Frazer and Fulco[43] neglected four-pion and higher states completely and put G(t) = 1, but although we have no way yet of calculating ζ_{in} it is well to remember that such a correction for multi-pion states exists. Our final result for the form factor is then

$$F_\pi(t) \approx [eD_1^1(0)/D_1^1(t)](1 + t/\zeta_{in}) \tag{15-12}$$

It is evident from (10-1) and (15-12) that if the P-wave π-π phase shift is small for $0 < \nu \lesssim 10$ (or $4 < t \lesssim 40$), then the pion-charge structure factor differs very little from e over a corresponding range of t along both positive and negative axes. In other words, the pion behaves almost as a point charge for electron scattering, and the cross section for π^+-π^- production in e^--e^+ annihilation is "normal." Suppose, on the other hand, that there is a reasonably sharp dynamical resonance at $t = t_R$, that is to say, the real part of the denominator function vanishes at t_R, with a nearly linear dependence extending down to t = 0† (see Fig. 15-2). Such a behavior is typical of resonances due to attractive "forces" inside a centrifugal barrier. At the same time, of course, the imaginary part of D_1^1 does not vanish. According to (10-6) and (10-9), we may expect a behavior

$$\text{Im } D_1^1 = -[\nu^3/(\nu + 1)]^{1/2}\,\Gamma(t)$$

where $\Gamma(t)$ is a slowly varying function if the important contributions from the left-hand cut in (10-9) are not too close. Thus, we have the rough formula

$$\frac{1}{e}F_\pi(t) \approx \frac{t_r}{t_r - t - i\Gamma[\nu^3/(\nu + 1)]^{1/2}\,\theta(t - 4)} \tag{15-13}$$

†There is of course a branch point at t = 4, but it produces a discontinuity only in second and higher derivatives of Re $D_1^1(t)$.

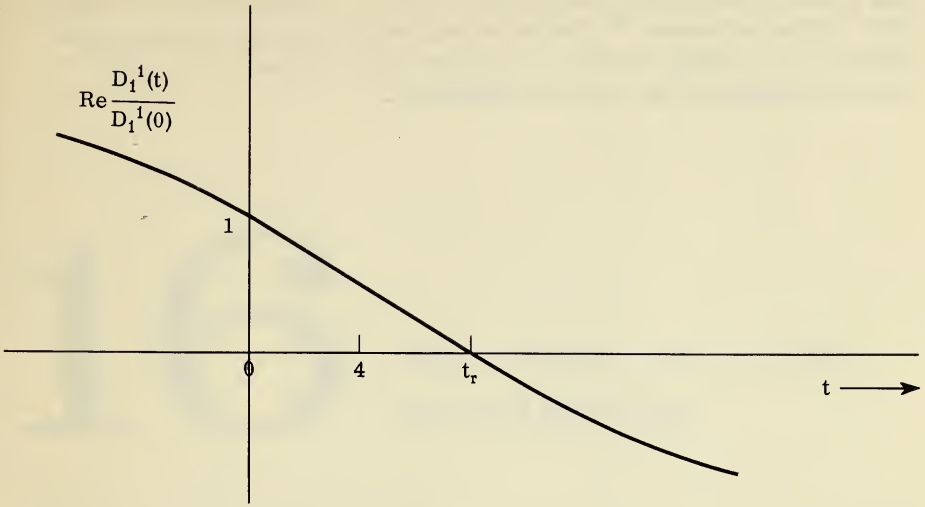

FIG. 15-2. Typical behavior of the real part of the denominator function for a dynamical resonance.

which can be used for $|t| \lesssim t_r$, $|t| \ll \zeta_{in}$.

In the electron-scattering region, $t < 0$, we see that the cross section is simply reduced by a smooth factor

$$\left[\frac{t_r}{t_r + 2q^2(1-\cos\theta)}\right]^2 \tag{15-14}$$

Comparing what we would get from a classical extended charge $\rho(\mathbf{r})$,

$$\begin{aligned} F_\pi &= \int d\mathbf{r}\, \exp[i(\mathbf{q}_1 - \mathbf{q}_2)\cdot\mathbf{r}]\rho(r) \\ &\approx e[1 - (\overline{r^2}/6)(\mathbf{q}_1 - \mathbf{q}_2)^2 + \cdots] \end{aligned} \tag{15-15}$$

we see that the "mean-square radius" of the pion is related to the resonance energy by

$$\overline{r^2}/6 \approx 1/t_r \tag{15-16}$$

A completely different kind of effect would be observed in the cross section for $e^+ + e^- \rightarrow \pi^+ + \pi^-$. There the "normal" cross section is multiplied by a resonance factor

$$\frac{t_r^2}{(t_r - t)^2 + \Gamma^2[\nu^3/(\nu+1)]} \tag{15-17}$$

that greatly enhances the reaction for t near t_r. Notice that for a fixed position of the resonance, t_r, the net enhancement becomes greater as the width of the resonance decreases because the maximum value of (15-17) varies inversely as the *square* of the width Γ.

16 THE PION-NUCLEON INTERACTION

Although more complicated than π-π scattering from a theoretical point of view there exists far more experimental information about the interaction of pions with nucleons. Let us then consider the general diagram of the type of Fig. 3-1, using the notation of Chapter 5, where there were introduced four invariant amplitudes $A^{0,1}$ and $B^{0,1}$ satisfying the Mandelstam representation. All three channels are interesting, but we start off the discussion by considering the direct and exchange "forces" acting in Channel III ($\pi N \rightarrow \pi N$) due to Channels II and I. Channel I is related to Channel III by crossing symmetry, but Channel II ($\pi\pi \rightarrow N\bar{N}$) is completely different and will require separate treatment.

The direct and exchange forces acting between a pion and a nucleon are not at all similar, as emphasized already in Chapter 8. The long-range direct forces coming from Channel II are due to the exchange of two pions, and formulas for the corresponding part of the generalized direct potential have been worked out by Singh and Udgaonkar.[44] Because of the spin complication these formulas are too lengthy to reproduce here, but they express the absorptive part for Channel II as an integral over the double-spectral functions given by the Cutkosky graphs of Fig. 16-1, with a single subtraction corresponding to the $J = 0$ component. In order to calculate the double-spectral functions of Fig. 16-1 it is seen that one needs the analytic continuation of both π-π and π-N scattering absorptive parts. A partial "bootstrap" mechanism is thus in operation. The computation of the $J = 0$ partial wave in Channel II presents a special problem that will be dealt with presently. Although a satisfactory calculation of these long-range direct π-N forces must await a solution of the asymptotic oscillation problem mentioned in Chapter 14, a number of rough estimates have been based on existing empirical knowledge about physical π-π and π-N absorptive parts.[45] A slightly surprising result is that in the low-energy elastic region the direct forces

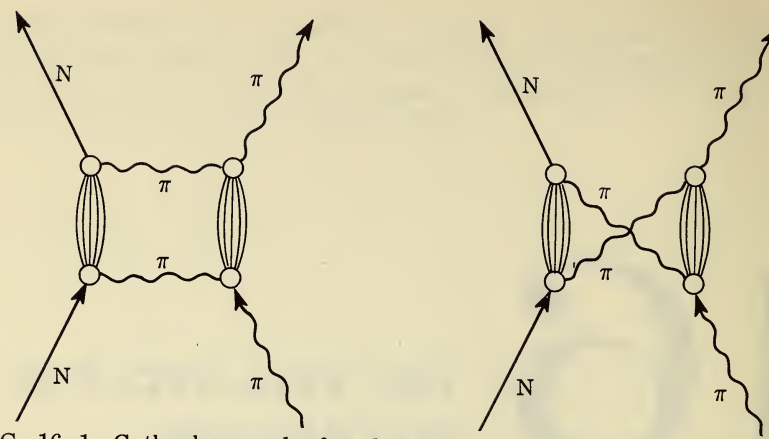

FIG. 16-1. Cutkosky graphs for the two-pion direct forces acting between a pion and a nucleon.

are relatively weak compared to the exchange forces still to be discussed. At high energies, however, the direct forces are sure to play a major role.

The longest-range part of the generalized exchange potential comes from the single-nucleon contribution to the absorptive part of Channel I as shown in Fig. 16-2(a), and is completely determined by the pion-nucleon coupling constant g_r^2. Actually the graph of Fig. 16-2(a) gives, at low energies, both a short-range and a long-range force because of the large difference in mass between the pion and the nucleon, which spoils the simple connection between force range and mass. This connection has its origin in the uncertainty principle and, as discussed in the introduction, if Fig. 16-2(a) is interpreted as a nucleon being exchanged from left to right, then the violation of energy conservation (or the reciprocal force range) turns out near threshold to be about $M^{1/2}$. On the other hand, if we say that an antinucleon goes from right to left, then the range is much shorter, about 1/2M.† It develops that the long-range part is very weak in S states of the π-N system but strong in P states. The short-range part is important mainly for S states but is repulsive, so it does not produce large phase shifts at low energy.[46]

The long-range part of the low-energy single-nucleon exchange force is contained—more or less correctly—in the static model of the pion-nucleon interaction studied in 1955 by Chew and Low.[6] This model was therefore able to make successful predictions about low-energy P-wave phase shifts. At the conclusion of this chapter we shall show how the modern approach can be used to derive the Chew-Low formula for the phase shift in the state with $J = 3/2$ and $T = 3/2$, where a strong attraction as shown in Fig. 16-2(a) makes possible a dynamical resonance. The corresponding forces in the other P states turn out to be repulsive.

†At very high energy this anomaly disappears and the range uniformly has the "normal" magnitude 1/M.

The next-longest-range generalized exchange potential is given by an integral over the double-spectral functions associated with the graphs of Fig. 16-2(b,c,d), with a subtraction of the $J = 1/2$ portion of the Channel I elastic absorptive part. The formulas here have been derived by Singh and Udgaonkar[44] and involve not only the absorptive parts for π-π and π-N scattering but also that for $\pi\pi \rightarrow N\bar{N}$. Estimates of these exchange forces[45] have shown them to be less important than single-nucleon exchange at low energies; at high energy their range is reduced to $\lesssim (M + 1)^{-1}$, so they cannot play a major role.

It was noted above that a special treatment of the $J = 0$ partial wave in Channel II ($\pi\pi \rightarrow N\bar{N}$) is required. In the following chapter, when we discuss

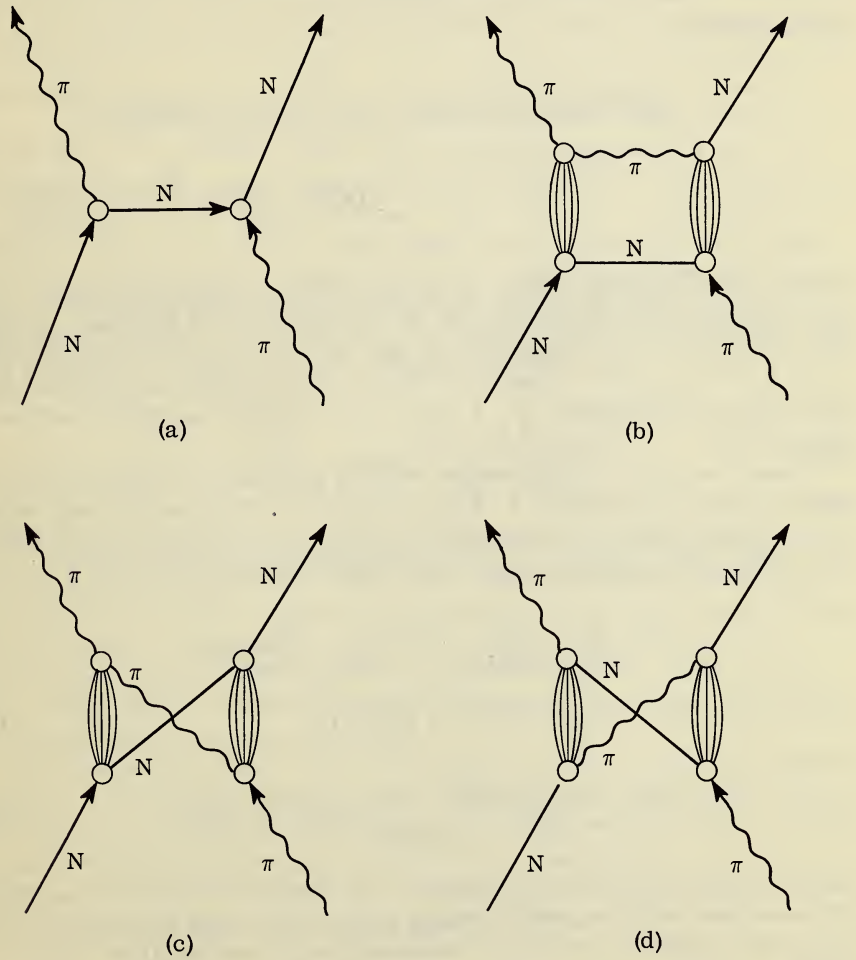

FIG. 16-2. Cutkosky graphs for the one- and two-particle exchange forces acting between a pion and a nucleon.

nucleon electromagnetic structure, the J = 1 partial waves also will be needed. Let us consider, therefore, the general problem of a partial wave in Channel II. Following Frazer and Fulco[47] we introduce the barycentric-system variables q_2, p_2 and $\cos\theta_2$, where q_2 is the magnitude of a pion momentum and p_2 that of a nucleon, while θ_2 is the production angle. We find

$$\begin{aligned} s_1 &= -{p_2}^2 - {q_2}^2 - 2p_2q_2\cos\theta_2 \\ s_2 &= 4({q_2}^2 + 1) = 4({p_2}^2 + M^2) \\ s_3 &= -{p_2}^2 - {q_2}^2 + 2p_2q_2\cos\theta_2 \end{aligned} \tag{16-1}$$

Next we define

$$[{A_1}^{II,T}(s_2);\ {B_1}^{II,T}(s_2)] = 1/2 \int_{-1}^{+1} d\cos\theta_2\, P_1(\cos\theta_2) \times [A^T(s_2, \cos\theta_2);\ B^T(s_2, \cos\theta_2)] \tag{16-2}$$

realizing that the index 1 here is not the orbital angular momentum. We know from crossing symmetry that A^0 and B^1 are even functions of $\cos\theta_2$ and A^1 and B^0 are odd functions, so that ${A_1}^{II,0}$ and ${B_1}^{II,1}$ vanish for odd l and ${A_1}^{II,1}$ and ${B_1}^{II,0}$ vanish for even l. It is also easy to show that near one of the two thresholds, $q_2 = 0$ and $p_2 = 0$, a partial amplitude behaves like ${q_2}^l$, ${p_2}^l$.

Frazer and Fulco then introduce helicity amplitudes, $f_{\pm}^J(s_2)$, for well-defined angular momentum J. The subscript (+) indicates that both nucleon and antinucleon have the same helicity, while (−) indicates opposite helicities. Using the technique of Jacob and Wick[48] they find, for each T,

$$f_+^J = \frac{1}{4\pi}\left\{-\frac{{p_2}^2}{(p_2q_2)^J}\,{A_J}^{II} + \frac{M}{(2J+1)(p_2q_2)^{J-1}} \times [(J+1)B_{J+1}^{II} + J\,B_{J-1}^{II}]\right\} \tag{16-3}$$

$$f_-^J = \frac{1}{4\pi}\frac{[J(J+1)]^{1/2}}{2J+1}\frac{1}{(p_2q_2)^{J-1}}(B_{J-1}^{II} - B_{J+1}^{II}) \tag{16-4}$$

and show that these helicity amplitudes are analytic functions of s_2, with a cut associated with Channel II running from 4 to ∞ along the positive real axis, and coincident cuts due to Channels I and III running from ${s_2}^L = 4 - 1/M^2$ to $-\infty$. Note that near the point $p_2 = 0$ the function f_+^0 behaves like ${p_2}^2$, and that there is no function f_-^0.

Problem: Show that a fixed value of s_3' in the spectrum of Channel III gives rise to a cut in the helicity amplitude running from

$$[4M^2 - (s_3' - M^2 - 1)^2]/s_3' \text{ to } -\infty$$

The discontinuity across the left-hand cut has two parts, one from the poles starting at s_2^L and one from the continuous π-N spectra starting at $s_2 = 0$. Frazer and Fulco give explicit formulas for this discontinuity in terms of g_r^2, A_3^T, and B_3^T, the absorptive parts of the pion-nucleon elastic-scattering amplitude. We shall not write down these formulas because of their complexity; they have basically the same structure as (9-8), except for the additional term proportional to g_r^2 that controls the left-hand cut for $0 < s_2 < s_2^L$.

Next Frazer and Fulco consider the right-hand cut and show that for $4 < s_2 < 16$ unitarity requires the helicity amplitudes to have the same phase as the π-π amplitude in the corresponding state (same J and T). The derivation of this result parallels the discussion of the preceding chapter. The next step is to argue, as before, that if we are concerned only with $|s_2| \lesssim 40$, we may use the simple phase condition for the entire right-hand cut; the helicity amplitudes may then be explicitly written down in terms of the discontinuity over the left-hand cut and the appropriate π-π denominator function. For example, if we assume no J = 0 elementary particles with the quantum numbers of Channel II, then

$$f_+^0(s_2) = \frac{p_2^2}{\pi D_0^0(s_2)} \int_{-\infty}^{s_2^L} ds_2' \frac{\text{Im } f_+^0(s_2')D_0^0(s_2')}{(s_2' - s_2)p_2'^2} \tag{16-5}$$

We see by inspection that this formula makes the helicity amplitude an analytic function with the two desired cuts, and at the same time assigns the correct phase on the right-hand cut and the correct discontinuity across the left-hand cut. The problem is therefore solved for J = 0 once the analytic continuations of the absorptive parts for Channels I and III are known. The higher J amplitudes should not require this special treatment, but in some cases it may be convenient.

Let us now take at least a brief look at some of the complications arising in a study of the Channel I and Channel III partial-wave amplitudes. Because of crossing symmetry it suffices to study Channel III, where we introduce the usual barycentric-system variables, q_3 and $\cos\theta_3$:

$$q_3^2 = \frac{[s_3 - (M+1)^2][s_3 - (M-1)^2]}{4s_3}$$

$$s_2 = -2q_3^2(1 - \cos\theta)$$

$$s_1 = \frac{1 - \cos\theta_3}{2}\,\frac{(M^2 - 1)^2}{s_3} - \frac{1 + \cos\theta_3}{2}\,(s_3 - 2M^2 - 2) \qquad (16\text{-}6)$$

These formulas already suggest the woe in store for us as a result of the unequal masses of pion and nucleon. Next we define $A_1^{III}(s_3)$ and $B_1^{III}(s_3)$ in the usual way and relate these quantities to amplitudes for well-defined J and parity. The conventional notation here is to write $f_{1\pm}^{III}$, where l is the orbital angular momentum and (±) refers to $J = l \pm 1/2$. In the physical region we have

$$f_{1\pm}^{III} = e^{i\delta_{1\pm}} \sin\delta_{1\pm}/q_3 \qquad (16\text{-}7)$$

Henceforth we shall drop the superscript III. The relation between A_l, B_l, and $f_{l\pm}$ turns out to be[25]

$$f_{1\pm}(W) = \frac{1}{8\pi W}\{(E + M)[A_1 + (W - M)B_1] + (E - M)$$

$$\times[-A_{1\pm1} + (W + M)B_{1\pm1}]\} \qquad (16\text{-}8)$$

where $W = (s_3)^{1/2}$ and

$$E = (q_3^2 + M^2)^{1/2} = (W^2 + M^2 - 1)/2W \qquad (16\text{-}9)$$

Examination of (16-8) shows that $f_{1\pm}$, as a function of s_3, has a "kinematical" branch point at the origin but that as a function of W all the singularities are of the usual "dynamical" type—that is, they arise from the vanishing of denominators in the original Mandelstam representation. McDowell[25] emphasized that an interesting reflection property exists in the W plane—which encompasses two sheets of the usual s_3 plane. We see from (16-9) that we have

$$f_{1+}(-W) = -f_{(1+1)-}(W) \qquad (16\text{-}10)$$

so if we work in the full W plane, the two amplitudes for the same J may be considered as a single analytic function in different halves of the plane. In view of this circumstance and for certain other reasons associated with the peculiarities of the kinematical relations (16-6) and too complicated to discuss here, Frazer and Fulco[34] introduce

$$h_1(W) = \frac{W}{E + M}\,\frac{f_{1+}(W)}{q_3{}^{21}} \qquad (16\text{-}11)$$

as the most convenient analytic function. For W real and greater than M + 1 this function is related to the phase shift for total angular momentum

J and orbital angular momentum $l = J - 1/2$, while for W real and less than $-(M + 1)$ we are dealing with the phase shift for the same J but $l = J + 1/2$.

The singularities of $h_l(W)$ are very complicated. First there are the two physical cuts, running from $M + 1$ to $+\infty$ and from $-(M + 1)$ to $-\infty$, both on the real axis. For $J = 1/2$ and $T = 1/2$, there is also the nucleon pole in h_0 on the left half-plane at $W = -M$. These are all the singularities arising from Channel III. The original pole in Channel I (also $\pi + N \leftrightarrow \pi + N$) leads to short branch cuts, on the positive real axis for $M - 1/M \leq W \leq (M^2 + 2)^{1/2}$ and on a corresponding range of the negative real axis. These cuts are so short that they are well approximated by poles. A second branch cut arising from the original Channel I pole runs along the entire length of imaginary axis. It has already been explained why the original nucleon pole leads in this way to both a long-range and a short-range force. The continuum singularities of Channel I, starting with the one-pion one-nucleon contributions, similarly lead to two cuts, and for the same reason. The "nearby" cuts run from $(M - 1)$ and $-(M - 1)$ to the origin, while the far cuts again run along the imaginary axis.

The intermediate states of Channel II turn out to produce a complicated cut in the W plane for the Channel III partial-wave amplitudes, partly on the imaginary W axis but also in part following a circle of radius M with center at the origin. The ends of this cut come close to the physical regions and correspond to the long-range direct forces due to 2π exchange.

Formulas for the discontinuities across all the various cuts have been worked out in terms of absorptive parts for appropriate channels,[34,45] and an extensive investigation of just how much can be understood about the observed phase shifts in terms of nearby and already calculable singularities has been undertaken by Frautschi and Walecka as well as others.[45,46] As mentioned already, the qualitative success of the static model has been to a certain extent understood. It has been shown, for example, that the π-π cut has only a weak effect in the $I = 3/2$, $J = 3/2$ state, as does the nearby part of the crossed π-N continuum cut, so the dominant nearby singularity in this state is the short cut, near $W = M$, which can be approximated by a pole of residue $\sim g_r^2$. Replacing all other singularities by a distant pole and proceeding as in our derivation of the n-p effective range formula in Chapter 10 then leads to the Chew-Low formula for δ_{33},[6]

$$\frac{4}{3} f^2 \frac{q_3^3}{W - M} \cot \delta_{33} \approx \frac{W_R - W}{W_R - M} \tag{16-12}$$

where $f^2 = (1/4M^2)g^2 \approx 0.08$ and W_R is the energy of the 3/2, 3/2 resonance. The value of W_R is related to the strength of the distant phenomenological pole and cannot be predicted until we have better calculational methods. However the width of the resonance is correctly predicted in terms of f^2, showing that the nucleon and the 3,3 resonance are not *both* elementary particles.

17 ELECTROMAGNETIC STRUCTURE OF THE NUCLEON

The basic methods of the S-matrix approach to strong interaction theory have now been outlined and several examples discussed. We conclude these lectures with an example of unusual importance from a practical standpoint. This is the problem of the nucleon-photon vertex, or in more familiar language, the electromagnetic structure of the nucleon. Recalling the considerations of Chapter 15, we realize that the nucleon-photon vertex function can be experimentally measured in the range $t < 0$ by electron-nucleon elastic scattering and for $t > 4M^2 \approx 180$ by $N + \overline{N} \longleftrightarrow e^+ + e^-$. The latter range is very high from the point of view of our approach; however, the cut in the photon-nucleon vertex function does not begin at $4M^2$ but rather at 4, the two-pion threshold. By the usual arguments, we should expect that the discontinuity across the cut for $4 < t \lesssim 40$ should dominate the behavior of the function for $|t| \lesssim 40$; thus it is reasonable to try to understand electron-nucleon scattering up to a few Bev electron energy (lab) in terms of two- and three-pion contributions to the spectral function of the photon-nucleon vertex.

Actually four invariant functions are involved in the photon-nucleon vertex, because of the nucleon spin and charge degrees of freedom. Using the same kind of notation as in (5-1), we would write[49]

$$\overline{u}_{p_2}\{i\,\gamma\cdot\epsilon\,[G_1{}^S(t) + \tau_3 G_1{}^V(t)] + \gamma\cdot\epsilon\,\gamma\cdot(p_1 - p_2)[G_2{}^S(t) + \tau_3 G_2{}^V(t)]\}\,u_{p_1} \qquad (17\text{-}1)$$

where ϵ is the photon polarization vector and where the superscripts S and V refer to the isotopic scalar and vector parts of the electromagnetic

interaction.† The vertex functions G(t) are related at t = 0 to the static nucleon charges and anomalous magnetic moments:

$$G_1^S(0) = G_1^V(0) = e/2 \tag{17-2a}$$

$$G_2^S(0) = (\mu_P + \mu_N)/2 = -0.06(e/2M) \tag{17-2b}$$

$$G_2^V(0) = (\mu_P - \mu_N)/2 = 1.83(e/2M) \tag{17-2c}$$

Each of the four functions G(t) has a spectral representation of the type (15-5) with the associated spectral functions given by a formula of the type (15-7), except that the π^+-π^- state is replaced by an N-$\bar{N}$ state, with T = 0 for the isotopic scalar functions and T = 1 for the vector functions. The other quantum numbers are the same as for the pion-photon vertex.

It then turns out[50] that only even-pion intermediate states contribute to the isotopic vector part of nucleon electromagnetic structure and only odd-pion states to the scalar part. Considering first the vector part, let us calculate the spectral functions in the range $4 < t < 16$ where only 2π states contribute. Evidently the result is a product of the pion-photon vertex function $F_\pi(t)$ and the amplitudes for $\pi + \pi \longleftrightarrow N + \bar{N}$ in the T = 1, J = 1 state. In the preceding chapter we saw that there are two such functions, which may be chosen to be the helicity amplitudes $f_\pm^1(t)$. Frazer and Fulco[43] found the result

$$g_1^V(t) = -F_\pi(t)\,\frac{(t/4-1)^{3/2}}{t^{1/2}}\,\frac{M}{M^2-(t/4)}\left[\frac{t}{4\,2^{1/2}\,M} f_-^1(t) - f_+^1(t)\right]^\dagger \tag{17-3}$$

$$g_2^V(t) = F_\pi(t)\,\frac{(t/4-1)^{3/2}}{t^{1/2}}\,\frac{1}{M^2-(t/4)}\left[\frac{M}{2^{1/2}} f_-^1(t) - f_+^1(t)\right]^\dagger$$

which may plausibly be used up to $t \sim 40$ if four-pion states behave as we expect.

A priori we do not know how important in the spectral function is the region above that where the 2π state dominates. Experimentally, however, the measured slope of the functions G(t) at t = 0 gives us the mean value of 1/t in the weight function g(t)/t.[51] Specifically, if no subtraction is necessary, so that we write

$$G_i^V(t) = \frac{1}{\pi}\int_4^\infty dt'\,\frac{g_i^V(t')}{t'-t} \tag{17-4}$$

†Assuming "minimal" electromagnetic interactions, a photon can be thought of as carrying either isotopic spin 0 (scalar) or isotopic spin 1 (vector), but nothing more complicated. This follows from the fact that electric charge and the z component of isotopic spin are linearly related.

then we have

$$\left(\frac{1}{t}\right)^{V}_{av,i} = \frac{[(d/dt)G_i{}^{V}(t)]_{t=0}}{G_i{}^{V}(0)} \tag{17-5}$$

There are substantial theoretical reasons for expecting no subtraction in $G_2{}^{S,V}(t)$ and weaker ones for $G_1{}^{S,V}(t)$. In any case, it is found experimentally that,[51] with (17-5) as a definition of $(1/t)_{av}$, we have

$$(1/t)_{av,1}{}^{V} \approx (1/t)_{av,2}{}^{V} \approx 1/16 \tag{17-6}$$

This circumstance suggests a dominant role for the 2π state.

Frazer and Fulco[43] proceeded to calculate $G_{1,2}{}^{V}(t)$ on the basis of (17-4) and (17-3), with various assumptions for the P-wave π-π phase shift, which controls $f_{\pm}{}^{1}(t)$ through the analogue of Eq. (16-5) as well as $F_\pi(t)$ through Eq. (15-12). They found, as had earlier been emphasized by Drell,[52] that with a small P-phase shift, the value of $G_2{}^{V}(0)$ (the vector anomalous moment) is too small by a factor of about five unless large contributions to (17-4) come from the unknown region $t' > 40$. In such a case, however, the experimental result (17-6) is puzzling. In order to achieve (17-6) Frazer and Fulco had to assume a resonance in the π-π P wave at $t_r \approx 16$, which greatly enhances this part of the spectrum.† The mechanism of enhancement is quickly seen by reference to (17-3) and the $J = 1$ analogue of (16-5) from which follows $g_2{}^{V}(t) \sim |F_\pi(t)|^2$, so the spectral functions have the behavior (15-17) in the neighborhood of the resonance. The average enhancement depends on the width parameter Γ; to achieve the experimental result (17-2b) it appears that a width $\Gamma \approx 0.4$ is required.

Thus, the large nucleon anomalous magnetic moment together with the large radius of this moment $[(1/t)_{av} = (1/6)\overline{r^2}]$ suggests a dynamical resonance in the P state of the π-π system, and in Chapter 14 we pointed out that such a resonance can come about through a "bootstrap" attractive force. Of course, if the resonance exists there must be other experimental manifestations; the cleanest suggested so far is the enhancement by the factor (15-17) of pion-pair production in electron-position annihilation. In general one would expect enhancement of this kind whenever pion pairs are produced, but usually there are other strongly interacting particles simultaneously present which confuse the situation. The other general manifestation is the enhancement of the unphysical singularities in various amplitudes due to pion pairs. The subject of this chapter, the nucleon-photon vertex, is a typical example and we remark now on the consequence of a P-wave π-π resonance for $G_1{}^{V}(t)$, the isovector charge-vertex function, which is pro-

†Bowcock, Cottingham, and Lurié have observed that a good fit to the experimental $G_2{}^{V}(t)$ implies a negative contribution of ~20 per cent from large values of t' in (17-4). They then estimate $t_r \approx 22$.[45]

foundly affected together with the magnetic-moment vertex from which the existence of the resonance was inferred.

If we accept (17-4) for $G_1^V(t)$ without a subtraction, then the experimental near equality of the vector charge and magnetic-moment radii of the nucleon is immediately a consequence of a reasonably sharp π-π resonance. That is, both $g_1^V(t)$ and $g_2^V(t)$ are proportional to $|F_\pi(t)|^2$, so that the average value of $1/t$ should be $\sim 1/t_r$ for both. One may ask if (17-2a) is consistent with no subtractions in $G_1^V(t)$; a tentatively affirmative answer has been given by Frazer and Fulco,[43] considering the uncertainty in certain aspects of our knowledge of the $\pi + \pi \longleftrightarrow N + \bar{N}$ helicity amplitudes $f_\pm^1(t)$. We have in the equality of isovector charge and magnetic radii, therefore, some confirmation of the resonance hypothesis.

We conclude with a brief mention of $G_{1,2}^S(t)$, the isotopic scalar part of the nucleon-photon vertex. Experimentally $G_2^S(t)$ is very small over the entire range of t studied, while $G_1^S(t) \approx G_1^V(t)$. The latter circumstance means a low average value of t in $g_1^S(t)$ and, with no subtractions, suggests a low-energy resonance or perhaps even a bound state for the three-pion system with quantum numbers $J = 1$, $I = 0$, and odd parity. That such a state may feel an unusually strong total attraction follows from the fact that all three pairs are in the $I = 1$ configuration,[53] where we have conjectured a strong attractive force. However a quantitative calculation obviously must await a development of techniques for handling the three-body problem. That the scalar charge radius is so close to the vector charge radius appears a coincidence from the present point of view. If it is not a coincidence, then the approach described in these lectures is deficient in some very basic respect.

18 SUMMARY AND CONCLUSION

An attempt has been made in these lectures to make plausible that the analytically continued and "saturated" S matrix, with the established symmetries, contains all of strong-interaction physics. In this extreme view the elementary-particle concept is absent, and there are no arbitrary dimensionless parameters and probably only one dimensional constant to establish the scale of masses. In performing calculations to test the theory, one starts with any singularity of the S matrix and attempts to find what other singularities must be present to achieve consistency with analyticity and a "saturated" unitarity condition. In principle, *all* other singularities, and thus the entire S matrix, should be determined, but in practice it is only the immediate neighborhood of the given singularity that can be handled with any accuracy. The equations to be solved are nonlinear and the number of degrees of freedom increases rapidly the farther one goes. Therefore the verification of the theory may be described as an operation of infiltration: One starts at a number of different points and works outward from each to see if the S matrix connects properly from one region to the next.

Most of the examples of such calculations described in these lectures have centered about the least massive strongly interacting system, the pion. The "nearest" singularities here correspond to two-pion states and the region of the S matrix under study is that which is controlled—in the classical sense—by long- and intermediate-range forces, owing to exchange of one or two pions. The simplest example considered was the attempt to understand the existence of certain unstable particles, with the quantum numbers of a pion pair, given only a knowledge of the existence and mass of the pion. These particles were described as two-pion "dynamical resonances" produced by an attractive force arising from two-pion exchange. It should be realized, however, that this description was merely a convenient way of

characterizing the approximation employed. The pion need not be thought of as more fundamental than these unstable particles.

Now suppose that there is qualitative but not quantitative success in this first attempt. That is, the equations of Chapters 12 and 13 lead to a correct prediction of the states that should have resonances and the positions and widths that are not wrong by more than, say, a factor two. We should feel encouraged about the basic approach and blame the errors on short-range forces, but could not proceed with confidence to the next singularity because the errors by then would be intolerable. At this point, however, we can use experimental evidence about the resonances as a fresh and accurate starting point that is much closer to our next goal than the 1π state where we started originally. Such a technique evidently can be used over and over again and in fact it already has been much used in connection with π-N and N-N scattering as well as with nucleon electromagnetic structure.

I do not think that this approach by infiltration can be criticized on philosophical grounds because (1) no part of the strong-interaction S matrix is supposed to be more fundamental than any other, and (2) confrontation of theory with experiment always has and always will involve approximation and be limited to the simplest situations. A theory is deemed successful not when it has passed all possible tests (which never happens), but when it has passed an "impressive" number and failed none. This remark leads me to the final question: How much of strong-interaction physics is it reasonable to expect the analytically continued and "saturated" S matrix to predict *in practice?*

My guess is that with fast computers we shall eventually be able to handle three-body as well as two-body problems, but nothing more complicated. On the basis of the pion mass, therefore, one may hope to understand any 3π resonances as well as the 2π states mentioned above. Given the nucleon mass, the pion-nucleon coupling constant should be calculable as well as the (3/2, 3/2) resonance position. It ought to be possible to understand the entire nucleon-nucleon low-energy situation, except possibly for the hard core. High-energy π-π, π-N, and N-N total and elastic cross sections should be predictable, as well as the gross features of inelastic processes.

It goes without saying that in the picture presented here, the existence of eight stable baryons and seven stable mesons is to be regarded as a dynamical accident, although one that seems not particularly improbable. Given the pion mass as the smallest dimensional quantity, one does not expect the spacing of levels for a given simple set of quantum numbers to be much smaller than this unit, but if it is larger the upper level will decay by pion emission. Thus the existence of any stable excited states for a particular simple set of quantum numbers is unlikely, but by chance an excited level might occasionally occur. The π, N, K, Λ, and Ξ are all ground states in this sense, the Σ the only stable excited state.

I expect that, given the K as well as the π and N masses, one will eventually be able to predict the masses and quantum numbers of the Λ, Σ,

and Ξ and most of the interactions of these particles. The really difficult and perhaps impossible task will be the calculation of one of the three masses, m_π, m_N, m_K, from the other two. In principle, only one mass should be independent, but it may be beyond human powers to check this point. If, on the other hand, a reasonable number of particles now regarded as elementary are successfully "explained" through the S matrix, then one might be willing to give the theory the benefit of the doubt.

APPENDIX

Forward-Direction Elastic-Scattering Dispersion Relations: The Pomeranchuk Theorem

A very special but practically important example of a one-dimensional dispersion relation occurs when the momentum transfer for elastic scattering channels is fixed at the value zero. For an amplitude such as that of Fig. A-1, for instance, if we set $s_2 = 0$ we are considering forward-direction scattering for both Channel I and Channel III. In this situation the unitarity relation (4-3) in the physical region for Channel III reduces to the "optical theorem"†

$$A_3(s_1,0,s_3) = \frac{q_3 s_3^{1/2}}{8\pi}\sigma_3^{\text{tot}}(s_3) \qquad s_3 > (m+M)^2 \tag{A-1}$$

with a corresponding formula for A_1. The relation (3-7b) then becomes

$$A(s_1,0,s_3) = \text{const.} + \frac{1}{\pi}\int_{(M+m)^2}^{} ds'\left[\frac{A_1(s',0,2M^2+2m^2-s')}{s'-s_1} + \frac{A_3(2M^2+2m^2-s',0,s')}{s'-s_3}\right]$$

$$+ \frac{1}{8\pi^2}\int_{(M+m)^2}^{\infty} ds' q'(s')^{1/2}\left[\frac{\sigma_1^{\text{tot}}(s')}{s'-s_1} + \frac{\sigma_3^{\text{tot}}(s')}{s'-s_3}\right] \tag{A-2}$$

†If spin is present, we are speaking of the non-spin-flip forward amplitude. For the π-N amplitudes of Chapter 5 this is $A + \nu B$.

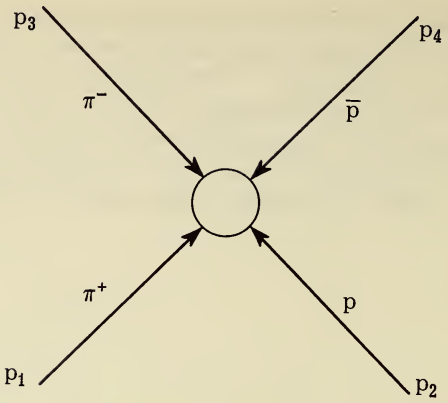

FIG. A-1. Diagram for the three channels

I. $\pi^- + p \rightarrow \pi^- + p$

II. $\pi^- + \pi^+ \rightarrow p + \bar{p}$

III. $\pi^+ + p \rightarrow \pi^+ + p$

The variables s_1 and s_3 are of course not independent in view of the supplementary condition (2-7), which here reduces to

$$s_1 + s_3 = 2M^2 + 2m^2 \tag{A-3}$$

It is conventional to introduce a new variable,

$$\nu = (s_1 - s_3)/4M \tag{A-4}$$

which may be verified as being equal to the "laboratory system energy" for Channel I and the negative of the lab energy for Channel III. (By "lab energy" we mean the total energy of the particle of mass m, when the particle of mass M is at rest.) If then $A(s_1,0,s_3)$ is written as $A(\nu)$ and variables are changed in (A-2), we find

$$A(\nu) = \text{const.} + \frac{1}{\pi}\int_m^\infty d\nu' \left[\frac{A_1(\nu')}{\nu' - \nu} + \frac{A_3(-\nu')}{\nu' + \nu}\right] + \frac{M}{8\pi^2}\int_m^\infty d\nu' (\nu'^2 - m^2)^{1/2}$$

$$\times \left[\frac{\sigma_1^{\text{tot}}(\nu')}{\nu' - \nu} + \frac{\sigma_3^{\text{tot}}(\nu')}{\nu' + \nu}\right] \tag{A-5}$$

since

$$q_1 \, s_1^{1/2} = (\nu^2 - m^2)^{1/2} M$$

If the least-massive pair of particles with the quantum numbers of Channels I and III is the pair under consideration, then the spectrum in (A-2)

below $(M + m)^2$ consists at most of poles, corresponding to single-particle states with appropriate quantum numbers. For π-π and π-K scattering there happen to be no such particles, so (A-5) with one subtraction (which we make for convenience at $\nu = 0$) becomes simply

$$A(\nu) = A(0) + \frac{M}{8\pi^2}\nu \int_m^\infty d\nu' \frac{(\nu'^2 - m^2)^{1/2}}{\nu'}\left[\frac{\sigma_1^{tot}(\nu')}{\nu' - \nu} - \frac{\sigma_3^{tot}(\nu')}{\nu' + \nu}\right] \quad \text{(A-6)}$$

One subtraction is sufficient if the total cross sections approach constants at infinity, because of the Pomeranchuk theorem[33] that

$$\lim_{\nu \to \infty} \sigma_1^{tot}(\nu) = \lim_{\nu \to \infty} \sigma_3^{tot}(\nu) \quad \text{(A-7)}$$

This theorem may be understood in a rough way by making a second subtraction in (A-6) and observing that

$$\lim_{\nu \to \infty} \text{Re } A(\nu) \sim \nu \ln \nu \left[\lim_{\nu \to \infty} \sigma_3^{tot}(\nu) - \lim_{\nu \to \infty} \sigma_1^{tot}(\nu)\right] \quad \text{(A-8)}$$

so that the ratio of Re A to Im A increases logarithmically unless (A-7) is fulfilled.

The formula (A-6) is remarkable in that only experimentally measurable quantities appear, apart from the one subtraction constant. In the case of π-N scattering, a pole corresponding to the single-nucleon state must also be included and, as discussed in Chapter 4, the residue of this pole is the pion-nucleon coupling constant. The substitution of experimental data into the forward-direction π-N dispersion relation thus leads to a determination of this extremely important constant.

Other systems for which sufficient data exist to apply the above considerations are the K-N and the N-N. Here, however, in addition to poles, one has nonphysical continua for $\nu < m$ which cannot be directly measured but must be obtained by some kind of analytic continuation. The conclusion from all investigations made to date of forward dispersion relations is that they are satisfied to within the accuracy of available experiments.

REFERENCES

[1]W. Heisenberg, Z. Physik, **120**, 513, 673 (1943).
[2]W. Heisenberg, Proceedings of the 1958 Annual International Conference on High Energy Physics at CERN, Geneva, 1958.
[3]L. D. Landau, Proceedings of the 1959 Annual International Conference on High Energy Physics at Kiev, Soviet Academy of Sciences, 1959.
[4]L. D. Landau, Nuclear Phys., **13**, 181 (1959) (included in this volume).
[5]R. E. Cutkosky, J. Math. Phys., **1**, 429 (1960) (included in this volume); also Phys. Rev. Letters, **4**, 624 (1960).
[6]G. F. Chew and F. E. Low, Phys. Rev., **101**, 1570 (1956).
[7]M. L. Goldberger, Phys. Rev., **99**, 979 (1955).
[8]R. Karplus and M. Ruderman, Phys. Rev., **98**, 771 (1955).
[9]R. Oehme, Phys. Rev., **100**, 1503 (1955); **102**, 1174 (1956); G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev., **106**, 1337 (1957).
[10]S. Mandelstam, Phys. Rev., **112**, 1344 (1958) (included in this volume); also Phys. Rev., **115**, 1741, 1752 (1959).
[11]G. F. Chew and S. C. Frautschi, Phys. Rev. Letters, **5**, 580 (1960), and Phys. Rev., **123**, 1478 (1961).
[12]C. Möller, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd., **23**, 1 (1945).
[13]T. Kibble, Phys. Rev., **117**, 1159 (1960) (included in this volume).
[14]See, for example, lecture notes by G. Källën and by R. Omnes, "Relations de dispersion et particules elementaires," Ecole d'été de physique théorique, Les Houches, 1960.
[15]R. J. Eden, Phys. Rev., **121**, 1567 (1961); P. V. Landshoff, J. C. Polkinghorne, and J C. Taylor, Nuovo cimento, **19**, 939 (1961).
[16]M. L. Goldberger, M. T. Grisaru, S. W. McDowell, and D. Wong, Phys. Rev., **120**, 2250 (1960).
[17]A. C. Hearn, "The Invariant Amplitudes of Interaction Processes," Cambridge University preprint, 1961.

[18]G. F. Chew, M. L. Goldberger, L. E. Low, and Y. Nambu, Phys. Rev., **106**, 1337 (1957).
[19]W. Frazer and J. Fulco, Phys. Rev., **117**, 1603 (1960).
[20]See the review by M. Moravscik and H. P. Noyes, Ann. Rev. Nuclear Sci., 11 (1961).
[21]G. F. Chew and S. C. Frautschi, Lawrence Radiation Laboratory Report UCRL-9685 (to be published in Phys. Rev.).
[22]R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Trieman, Ann. Phys., **10**, 62 (1960) (included in this volume).
[23]J. Charap and S. Fubini, Nuovo cimento, **14**, 540 (1959).
[24]G. F. Chew and S. C. Frautschi, Phys. Rev. Letters, **5**, 580 (1960).
[25]S. W. McDowell, Phys. Rev., **116**, 774 (1960).
[26]G. F. Chew and S. Mandelstam, Phys. Rev., **119**, 467 (1960).
[27]M. Froissart (to be published in Nuovo cimento).
[28]R. Omnes, Nuovo cimento, **8**, 316 (1958); **21**, 524 (1961).
[29]L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev., **101**, 453 (1956).
[30]G. F. Chew and S. Mandelstam, Nuovo cimento, **19**, 752 (1961).
[31]H. P. Noyes and D. Wong, Phys. Rev. Letters, **3**, 191 (1959).
[32]M. Froissart, Phys. Rev., **123**, 1053 (1961).
[33]S. Pomeranchuk, J. Exptl. Theoret. Phys. (U.S.S.R.), **34**, 725 (1958), translation: Soviet Phys JETP, **7**, 499 (1958).
[34]W. Frazer and J. Fulco, Phys. Rev., **119**, 1420 (1960).
[35]T. Regge, Nuovo cimento, **14**, 951 (1959); **18**, 947 (1960) (included in this volume).
[36]M. Froissart, International Conference on the Theory of Weak and Strong Interactions, La Jolla, 1961.
[37]V. N. Gribov, Nuclear Phys., **22**, 249 (1961).
[38]G. F. Chew, S. Mandelstam, and H. P. Noyes, Phys. Rev., **119**, 478 (1960).
[39]G. F. Chew and S. Mandelstam, Nuovo cimento, **19**, 752 (1961).
[40]Proceedings of the Berkeley Conference on Strong Interactions, Revs. Modern Phys., **33**, 355 (1961).
[41]B. R. Desai, Phys. Rev. Letters, **6**, 497 (1961).
[42]R. Karplus, C. Sommerfield, and E. Wichmann, Phys. Rev., **111**, 1187 (1958).
[43]W. Frazer and J. Fulco, Phys. Rev. Letters, **2**, 364 (1959); Phys. Rev., **117**, 1609 (1960).
[44]V. Singh and B. M. Udgaonkar, Phys. Rev., **123**, 1487 (1961).
[45]S. Frautschi and D. Walecka, Phys. Rev., **120**, 1486 (1960); J. Bowcock, W. Cottingham, and D. Lurié, Phys. Rev. Letters, **5**, 386 (1960).
[46]J. Uretsky, Phys. Rev., **123**, 1459 (1961).
[47]W. Frazer and J. Fulco, Phys. Rev., **117**, 1603 (1960).
[48]M. Jacob and G. C. Wick, Ann. Phys. **7**, 404 (1959).

[49]M. Levy, G. Ravenhall, and D. R. Yennie, Revs. Modern Phys., **29**, 144 (1957).
[50]G. F. Chew, R. Karplus, S. Gasiorowicz, and F Zachariasen, Phys. Rev., **110**, 265 (1958); P. Federbush, M. L. Goldberger, and S. B. Trieman, Phys. Rev., **112**, 642 (1958).
[51]R. Hofstadter and R. Herman, Phys. Rev. Letters, **6**, 293 (1961).
[52]S. D. Drell, Proceedings of the 1958 Annual International Conference on High Energy Physics at CERN, Geneva, 1958.
[53]G. F. Chew, Phys. Rev. Letters, **4**, 152 (1960).

7.A

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ON ANALYTIC PROPERTIES OF VERTEX PARTS IN QUANTUM FIELD THEORY

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Abstract: A general method of finding the singularities of quantum field theory values on the basis of graph techniques is evolved.

1. Introduction

In recent years many papers have been concerned with dispersion relations. As is known, the latter express the analytic properties of various quantities of quantum field theory. The problem of localizing the singularities of these quantities is therefore highly important. As has become clear recently [1, 2]), a direct study of graphs is the most effective method of investigating the location and nature of the singularities of vertex parts. It is often claimed that the graph technique is not sufficiently convincing, since it involves perturbation theory, unlike other, allegedly more rigorous methods. This view is actually based on misunderstanding. Since a rigorous theory which makes use of the Hamiltonian reduces interaction to zero, the sole completely rigorous dispersion relation in this theory is $0 = 0$. By posing the problem of analytic properties of quantum field values, we actually go beyond the framework of the current theory. An assumption is thereby automatically made that there exists a non-vanishing theory in which ψ-operators and Hamiltonians are not employed, yet graph techniques are retained. In evolving dispersion relations, therefore, the employment of the graph technique is, indeed, solely consistent, since the problem becomes meaningless if the graph technique is rejected.

The graph technique is by no means equivalent to perturbation theory, since all particles, stable as regards strong interactions, are considered, whether they are "simple" or "complex". In fact, first steps are thereby taken towards new graph techniques, which will be a generalization of the previous methods and will lay the foundation of the future theory. It stands to reason that the applicability of the graph technique of this type is likewise hypothetical and a test of successive results will be a test of the hypothesis itself.

Unfortunately, expressions obtained in considering more complex graphs are very lengthy. It may be demonstrated, however, that a study may be carried out in general form and be simplified considerably with the help of a corresponding graphic representation.

2. General Method

An arbitrary graph represents a certain integral

$$\int \frac{B\mathrm{d}^4k\mathrm{d}^4l\ldots}{A_1A_2A_3\ldots} \tag{1}$$

where

$$A_i = m_i^2 - q_i^2, \tag{2}$$

q_i is a certain 4-momentum, corresponding to the given line in a graph, m_i the mass of the particle in question and B a certain polynomial of the vectors q_i. According to the well known Feynman method, one may write

$$\frac{1}{A_1A_2A_3\ldots} = (n-1)!\int_0^1\ldots\int_0^1 \frac{\mathrm{d}\alpha_1\mathrm{d}\alpha_2\ldots\mathrm{d}\alpha_n\,\delta(\alpha_1+\alpha_2+\ldots+\alpha_n-1)}{(\alpha_1A_1+\alpha_2A_2+\ldots+\alpha_nA_n)^n}. \tag{3}$$

The expression $\alpha_1A_1+\alpha_2A_2+\ldots$ in the denominator is a polynomial of second degree with respect to the integration variables $k, l, \ldots$ The terms linear with respect to $k, l, \ldots$ may always be eliminated by transforming integration variables, after which we obtain

$$\alpha_1A_1+\alpha_2A_2+\ldots = \varphi+K(k', l', \ldots). \tag{4}$$

Here K is a homogeneous quadratic form of the new integration variables with coefficients depending only on the parameters α_i, and φ is a heterogeneous quadratic form of the vectors p_i, describing the free ends of the graph under consideration.

Let us confine ourselves to the case of real values of the squares and the scalar products of the vectors p_i. It is easy to verify that if the values of φ are positive the integral with respect to $k, l, \ldots$ is a real quantity (a self-conjugated spinor in the case of spinor functions), since the quadratic form K becomes positive definite (because all α_i are positive) when $k_4 \to i\kappa$, $l \to i\lambda$ are substituted. The vertex part is real, therefore, if $\varphi > 0$ for all values α_i; if, on the other hand, $\varphi < 0$ for certain values α_i, the vertex part becomes complex. The nearest singularity of the vertex part is obviously located at the values p_i, for which φ vanishes for determined values of all α_i and is positive at all other α_i; in other words, the singularities correspond to the vanishing of the minimum value of φ, treated as a function of α_i. If the singularities in the complex region are meant, any extremum of the function φ should be considered. It will be noted that since φ is a homogeneous function of the first order with respect to the variables α_i, the condition $\sum\alpha_i = 1$

may be excluded when the conditions of the existence of the vanishing extremum are being found.

Let us denote $\alpha_1 A_1 + \alpha_2 A_2 + \ldots$ as f. Since K is a quadratic form of the variables k', l', . . ., it is clear that φ is the value of the function f at additional conditions

$$\partial f/\partial k' = \partial f/\partial l' = \ldots = 0,$$

and, since k differs from k' by a constant vector, the latter conditions may be written

$$\partial f/\partial k = \partial f/\partial l = \ldots = 0. \tag{5}$$

The positiveness of the values α_i should be taken into account in finding the conditions of the minimum. Hence it follows that either the condition $\partial\varphi/\partial\alpha_i = 0$ or $\alpha_i = 0$ should take place for every value α_i. In the latter case $\partial\varphi/\partial\alpha_i = 0$ should exist for the nearest singularity. According to the definition we have

$$\frac{\partial\varphi}{\partial\alpha_i} = \frac{\partial f}{\partial\alpha_i} + \frac{\partial f}{\partial k}\frac{\partial k}{\partial\alpha_i} + \frac{\partial f}{\partial l}\frac{\partial l}{\partial\alpha_i} + \cdots$$

Since all $\partial f/\partial k$ equal zero according to (5), the condition $\partial\varphi/\partial\alpha_i = 0$ is equivalent to the condition $\partial f/\partial\alpha_i = 0$, i.e., according to the definition of f,

$$A_i = 0. \tag{6}$$

Thus, the singularity of the vertex part may be obtained through a joint solution of the equations $A_i = 0$ (or $\alpha_i = 0$) with additional conditions

$$\sum_i \alpha_i \frac{\partial A_i}{\partial k} = \sum_i \alpha_i \frac{\partial A_i}{\partial l} = \ldots = 0. \tag{7}$$

It is essential that these equations should have solutions with positive α_i.

Thus, it may be claimed with respect to every line of the Feynman graph that it either satisfies the condition $q_i^2 = m_i^2$ or passes out of consideration altogether (when $\alpha_i = 0$). In the latter case the singularity under consideration may be ascribed not to the graph in question but to a graph from which the i-line is absent, i.e., the vertices connected by it are fused. In analysing the singularities of the graph, it is sufficient therefore to consider the case when all $\alpha_i \neq 0$.

It can easily be seen that the condition (7) may be written as $\sum \alpha_i q_i = 0$, where summation is carried out not with respect to all lines of the Feynman graph, but with respect to the aggregates of lines forming closed contours, the direction of the vector q_i corresponding to the direction of the contour. The positiveness of the coefficients α_i denotes that if the vectors q_i are regarded as the directions of forces, the possibility of a solution of eq. (7) denotes the possibility of picking such values of these forces as would enable them to preserve equilibrium.

3. Graphs for Green's Functions

It is comparatively easy to interpret the singularities as they arise with the help of the present method. Let us begin with considering the graphs for Green's functions. In this case all vectors q_i obtained from the indicated equations are obviously parallel to the same vector p.

3.1. VERTEX PART WITH TWO EXTERNAL ENDS

Consider the graph in fig. 1. Setting down the formula $\sum \alpha_i q_i = 0$ for the contour formed by any couple of lines and taking into account the positiveness of the quantities α_i, we may infer that all vectors q_i are equally directed with respect to the vertices of the graphs. Noting that the vector lengths are equal to the corresponding masses, we easily obtain the obvious result $p^2 = (\sum m_i)^2$ for the singularities. It will be noted that in the case of Green's functions the consideration of any other graphs, for example graphs of the type shown in fig. 2, is superfluous. The number of equations for determining

Fig. 1.

Fig. 2.

Fig. 3.

the values α_i is, indeed, equal to the number of independent contours in the graph, which is 2 in the example under discussion. Yet, the total number of values α_i equals 5 in this example; hence it is clear that one of them may be taken to equal zero, as a result of which this graph reduces to one of those considered above. It should be mentioned that these arguments likewise hold good for the graphs of the type shown in fig. 3, where a sum of momenta acts as a momentum.

3.2. SIMPLE VERTEX PART WITH THREE EXTERNAL ENDS

Let us proceed to a vertex part with three external ends. Since these represent three vectors lying in the same plane, according to the conservation laws, and the vectors k, l, ... may be obtained from (6) and (7), it is clear that the latter vectors lie in the same plane. Thus, the problem is reduced to a flat system of vectors.

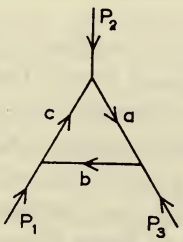

Fig. 4.

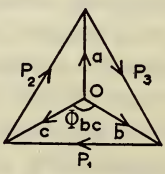

Fig. 5.

First of all, let us consider the simplest example, analysed in detail by R. Karplus, C. M. Sommerfield and E. H. Wichman [3]), as shown in fig. 4. It can easily be seen that the relation between vectors is represented by the scheme of fig. 5. The condition (7) obviously requires the point O to lie inside the triangle. It is implied that all vectors possess the properties of euclidean (not pseudo-euclidean) vectors, which can easily be proved for the nearest singularities.

Preliminarily, let us show how to express analytically the corresponding relations. First consider the scheme of fig. 5 which corresponds to the graph of fig. 4. If unit vectors $n_a = q_a/m_a$, $n_b = q_b/m_b$, $n_c = q_c/m_c$ are introduced, the condition (7) may be written as

$$\beta_a n_a + \beta_b n_b + \beta_c n_c = 0, \tag{8}$$

where $\beta_i = \alpha_i m_i$ are obviously positive quantities also. Projecting these equations on the vectors n_a, n_b, n_c consecutively and introducing the notations $(n_a n_b) = \mu_c = \cos \varphi_{ab}$ etc., we obtain three equations:

$$\begin{aligned} \beta_a + \beta_b \mu_c + \beta_c \mu_b &= 0, \\ \beta_a \mu_c + \beta_b + \beta_c \mu_a &= 0, \\ \beta_a \mu_b + \beta_b \mu_a + \beta_c &= 0. \end{aligned} \tag{9}$$

Making the determinant of this system equal to zero, we obtain the equation

$$1 + 2\mu_a \mu_b \mu_c = \mu_a^2 + \mu_b^2 + \mu_c^2, \tag{10}$$

which determines the situation of a singularity (if all β are simultaneously positive); μ_a, μ_b, μ_c are connected with p_1^2, p_2^2, p_3^2 by the formulae

$$\mu_a = \frac{m_b^2 + m_c^2 - p_1^2}{2m_b m_c}, \quad \mu_b = \frac{m_a^2 + m_c^2 - p_2^2}{2m_a m_c}, \quad \mu_c = \frac{m_a^2 + m_b^2 - p_3^2}{2m_a m_b}. \tag{11}$$

To return to the problem of the properties of vectors for the nearest singularities. Consider any angle, say φ_{bc}. According to (11) μ_a is a real quantity. It is greater than -1, since otherwise $p^2 > (m_b + m_c)^2$, i.e., we

Fig. 6.

would transcend the singularity determined by the graph in fig. 6, which is obtained from the graph under consideration through the elimination of one of the lines. At the same time we can see from (9) that no less than two out of three cosines of the angles should be negative. Now, a real angle corresponds to a negative cosine greater than -1. At least two of the three angles, therefore, are real. Since the sum of them equals 2π, the third angle is real as

well. Thus, fig. 5, which was mentioned in ref.[2]), solves the problem completely.

3.3. MORE COMPLEX GRAPHS WITH THREE EXTERNAL ENDS

Let us turn to more complex graphs. As an example, consider the graph shown in fig. 7. The scheme in fig. 8 corresponds to this graph. The construc-

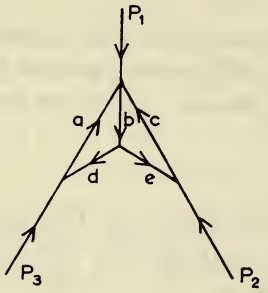

Fig. 7.

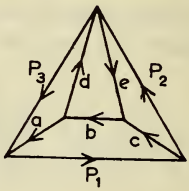

Fig. 8.

tion of this scheme is simple enough since every point in the Feynman graph corresponds to a polygon in the scheme with a number of sides equal to that of the rays converging at this point, while every polygon in the Feynman graph corresponds to a point of the scheme. The condition (7) calls for a certain disposition of the vectors a, b and d on the one hand, and b, c and e, on the other. It is not difficult to write an analytic expression of this figure. The triangle bde, as all other parts of the figure, is real, since the graph comprises only stable particles, and the mass of every particle in b, d, e is, therefore, less than the sum of the other two.

If one or several coefficients $\alpha_i = 0$ the graph is simplified. If, for example, $\alpha_c = 0$ the graph of fig. 7 will be reduced to the graph of fig. 9, similar to fig. 4.

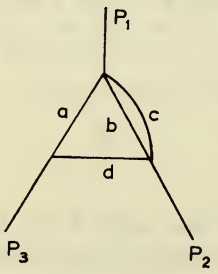

Fig. 9.

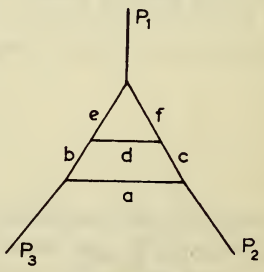

Fig. 10.

As we proceed to more complex Feynman graphs, it becomes apparent that often the case when all $\alpha_i = 0$ is impossible. Let us consider, for exam-

ple, the graph in fig. 10. The number of vector equations for α_i equals two in this case, i.e., we have four equations because the vectors lie in the same plane, while the number of the quantities α_i is six and therefore one of them may be assumed to equal zero. The same applies to the graph of the type shown in fig. 11.

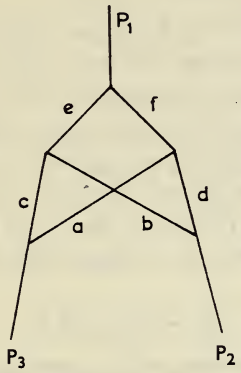

Fig. 11.

It can easily be seen how the above results change when what is meant is not the nearest singularities, that is, the emergence of complex values, but singularities of quantities that are already complex. If the singularity is not the nearest, we cannot, generally speaking, regard all angles as real, as was the case for the nearest singularities considered above.

First of all, consider the scheme of fig. 5 which corresponds to the graph of fig. 4. In the "euclidean" case with all the angles real, all μ_i lie between -1 and $+1$. In the general case a study of eq. (10) shows that the corresponding surface in the space μ_a, μ_b, μ_c consists of four parts intersecting each other at three points: $\mu_a = 1$, $\mu_b = -1$, $\mu_c = -1$, and the points obtained through the permutations of the indices a, b, c. One of these surfaces is limited by the triangle whose apices are the three points mentioned above. This triangle corresponds to the "euclidean" case. The three other surfaces extend to infinity, each starting from the corresponding point of intersection. On each of these surfaces one of the μ's is positive and exceeds 1, while the other two are negative and also exceed 1 in absolute value. As was pointed out, these surfaces always correspond to the non-nearest singular points.

It can easily be seen that the non-euclidean case is possible exclusively in the graph of fig. 4 out of all graphs considered above. In the scheme of fig. 8, for example, which corresponds to the graph of fig. 7, the vectors a, b, c on the one hand and a, b, d on the other, play the role of the vectors b, c, e. But, as was pointed out, the angles of the triangle formed by the vectors

b, d, e are always real. Otherwise, one of the particles b, d, e would not be stable. Yet, as was just mentioned, all cosines in the non-euclidean case should exceed unity in absolute value and consequently all angles should be complex.

3.4. GRAPHS WITH FOUR EXTERNAL ENDS

Let us proceed to the Feynman graphs with four ends. In this case it is obvious that we have to consider schemes which are located not in a plane but in space, which, naturally, complicates the problem a great deal. The most important of such graphs are the graphs with physical ends, i.e., those in which the squares of the corresponding momenta equal the squares of the particle masses. The simplest singularities are connected with the graphs of the type shown in fig. 12. The mean line may here correspond to a single particle or to several of them. The singularities of such a graph obviously correspond to

$$(p_1+p_2)^2 = m_a^2,$$

where m_a is the sum of the masses of the corresponding particles. The case of one particle yields an isolated pole, the case of two particles gives a singular line in the complex region.

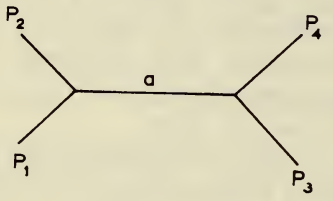

Fig. 12.

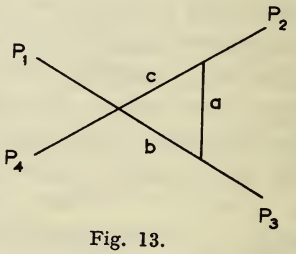

Fig. 13.

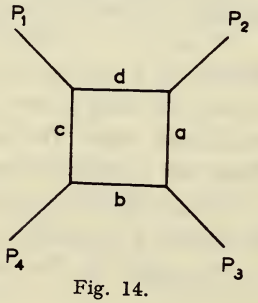

Fig. 14.

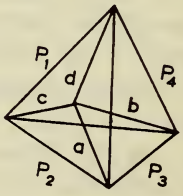

Fig. 15.

The graph in fig. 13 fully corresponds to the graph in fig. 4 and therefore no additional discussion is necessary.

The graph in fig. 14 corresponds to the scheme in fig. 15 where four of the six edges of the tetrahedron are determined by the length of free ends, while the two others equal p_1+p_2 and p_2+p_3 respectively. (It should be noted

that the graphic construction of this graph can be found in the paper of Karplus *et al.* [3]).) A detailed analysis shows that here as well, if the nearest singularity is discussed, all angles are real and the condition that the central point should be located within the tetrahedron corresponds to the condition of the positiveness of the coefficients α. A more complex situation, which will not be discussed in the present paper, corresponds to the singularities in the complex region. Just as was done in the analysis of the three-end graphs, it is convenient to introduce four unit vectors in the directions of the vectors a, b, c, d and find an equation connecting the cosines of the angles between them. An equation of the fourth degree arises instead of eq. (10). The following fact is worth mentioning. If graphs with four physical ends are considered, the following condition should be fulfilled for the graph of fig. 13, as is clear from the scheme of fig. 5: the sum of the angle of the triangle abp_3 between the sides a and b, and the angle of the triangle acp_2 between the sides a and c, both of which are fixed by the masses of the corresponding particles, should exceed π. An analogous condition for the graph of fig. 14 evidently is that the sum of the four angles in the triangles, corresponding to the four vertices, should exceed 2π. Hence, it directly follows that a necessary, though certainly not sufficient condition, for the existence of the "non-trivial" nearest singularities is the presence of an obtuse angle in at least one of the triangles, which, in its turn, requires that the virtual decay of at least one particle should be described by an inequality of the type

$$m_2^2 > m_a^2 + m_c^2.$$

This relation obviously cannot be fulfilled either for π-mesons or for nucleons. The nearest singularities in the scattering of these particles on each other, therefore, correspond to the graph of fig. 12. It should be emphasized, however, that this does not apply to the singularities in the complex region.

4. Nature of Singularities

Now we shall consider the nature of the singularities thus obtained. Let us return to the basic formula (1), writing it as

$$\int (\varphi + K)^{-n} B \mathrm{d}^4k' \mathrm{d}^4l' \ldots \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \ldots \delta(\alpha_1 + \ldots + \alpha_n - 1).$$

Expanding φ with respect to the powers $\alpha'_i = \alpha_i - \alpha_{0i}$ where a_{0i} corresponds to the minimum value of φ, we can write this integral

$$\int (\varphi_0 + Q)^{-n} B \mathrm{d}^4k' \mathrm{d}^4l' \ldots \mathrm{d}\alpha'_1 \mathrm{d}\alpha'_2 \ldots \delta(\alpha'_1 + \ldots + \alpha'_n), \qquad (12)$$

where φ_0 is the minimum value of φ_0 equal to zero in the singularity itself (the values of the vectors at the external ends being given), while Q is a quadratic function of the variables k', l', ... and the variables α'_i. To

elucidate the nature of the singularity, it is sufficient to consider the values B at $k' = l' = \ldots = 0$. If the power of the numerator with respect to the integration variables is lower than the power of the expression Q^n, the integral (12) converges at large values of these variables; in other words, its value is determined by the values of the variables corresponding to $Q \approx \varphi_0$, i.e., integration is carried out at small values of the variables, for which (12) has sufficient accuracy. It is evident that in this case (12) may be written as

$$\text{const.} \cdot \varphi_0^{\frac{1}{2}m-n} \tag{13}$$

where m is a number of integrations. If $m \geqq 2n$, the integral (12) does not converge and these arguments do not apply. To determine the nature of a singularity in this case the simplest way is to differentiate (12) with respect to φ_0 as many times as is necessary to make the power of the denominator greater than that of the numerator. After that we can make use of the formula obtained which should be integrated as many times. The integration constants obviously yield integer powers φ_0 which have no singularity at $\varphi_0 = 0$. Accordingly, we again obtain formula (13), with the exception of the case when $\frac{1}{2}m-n$ is zero or a positive integer. In this case, instead of (12) we obviously obtain

$$\text{const.} \cdot \varphi_0^{\frac{1}{2}m-n} \cdot \ln \varphi_0.$$

It should be noted that though the minimum is now discussed, these results may be applied to any extrenum φ as well.

The quantity n equals the number of internal lines in the Feynman graph; the number of vectors k with respect to which integration is carried out, equals the number ν of independent contours which make up the graph under consideration. Accordingly,

$$m = 4\nu+n-1.$$

Hence it follows that the nature of a singularity is determined by the expression $\varphi_0^{2\nu-\frac{1}{2}(n+1)}$ and, if $2\nu-\frac{1}{2}(n+1)$ is zero or a positive integer, by

$$\varphi_0^{2\nu-\frac{1}{2}(n+1)} \ln \varphi_0.$$

The quantity φ_0 is obviously proportional to the distance between the point under consideration and the hypersurface in the space p_i^2, $(p_i+p_k)^2$, in which singular points are situated. The number of contours should also include "biangles" which emerge when not one but several particles are conveyed along one line. For instance, $\nu = 2$ in the graph of fig. 9, and since $n = 4$, the singularity is of the nature of $\varphi_0^{\frac{3}{2}}$.

It will be noted that a more convenient number of vertices may be used than the number of independent contours. The number of independent contours, i.e., the number of independent integrations, equals the number of lines minus the number of additional conditions. The latter equals the

number of vertices less one, since one δ-function goes over into a final answer. Thus,

$$v = n - v + 1$$

(where v is a number of vertices) † and the power in the singularity may be written as $\frac{3}{2}(n+1)-2v$.

5. Summary

Let us formulate in brief the general rules for finding singularities. Various graphs with given external ends are considered. An arbitrary number of lines may converge at each vertex of such graphs, in agreement, of course, with the conservation laws (an odd number of π-meson lines cnnot pass through one point, for example). All particle stable in strong interactions may figure as lines. After that, a study is made of the scheme built on the principle of replacing the polygons in the graph by a scheme of vertices. The lengths of all internal lines in the scheme are equal to the corresponding masses. The essential intersections in the schemes (such as are obtained from the polygons in the graph) should satisfy the relation $\sum \alpha_i q_i = 0$, where q_i are the vectors proceeding from this intersection, and all $\alpha_i > 0$. The nearest singularities correspond to a scheme in euclidean space.

6. Application to Scattering Amplitude

A number of facts should be borne in mind in applying the results thus obtained to the scattering amplitude. Let us consider the scattering amplitude as a function of one variable x (which may be, for example, total energy or transferred momentum), regarding all other variables as given. The integrals considered above determine the functions whose values in the upper and lower semi-planes are connected by the relation $f(x^*) = f^*(x)$; in other words, the functions which are considered above and below the axis are actually different functions which are by no means analytic continuations of each other, and, generally speaking, we have a discontinuity if the values x are real. The scattering amplitude which is obtained when the Feynman rules of direction are observed, is in general of the form $a(x+i\delta)+b(x-i\delta)$, where δ is infinitesimal. The analytic properties considered above refer to the function $a(x+i\delta)$, extended by the vriable x into the upper semi-plane, and to the function $b(x-i\delta)$, extended into the lower semi-plane. The behaviour of the analytic continuation of the function $a(x+i\delta)$ into the lower semi-plane and of the function $b(x-i\delta)$ into the upper semi-plane by no means follows from the above.

† This was suggested to me by L. Okun and A. Rudik.

In an "alien" semi-plane the functions a and b may have any singularities located in any manner. Nor can they be determined from any general considerations. As is well-known, the proton-neutron scattering amplitude has, for example, apart from the singularity connected with the formation of a deuteron, another singularity connected with the virtual state of this system, which does not correspond to any real particle and lies precisely in an "alien" semi-plane with respect to the total energy of the system. Another example is the well-known resonance in the scattering of π-mesons on nucleons, which also corresponds to a singularity in an "alien" semi-plane, the total energy of the system being obviously complex. It is clear that such singularities cannot in principle be anticipated from general considerations, but may be obtained only from a theory yielding specific expressions for the scattering amplitude.

The problem is very much simplified in the case when there exists a region of values of x, for which the amplitude in question is real. Then we have two cuts through the real axis in the complex plane, and it is easy to see that the function $a(x+i\delta)$ has only the right cut, while the function $b(x-i\delta)$ has only the left one. If the quantity $a+b^*$ is considered instead of the scattering amplitude, for which it is sufficient to change the sign of the imaginary part of the amplitude in the left cut, we shall obtain a function which has no singularities in the upper semi-plane, which leads to the conventional dispersion relations.

The author is indebted to L. B. Okun, A. P. Rudik and Ya. A. Smorodinsky for numerous valuable comments.

References

1) X. Nambu, Nuovo Cimento **6** (1957) 1064
2) R. Karplus, C. M. Sommerfield and E. H. Wichman, Phys. Rev. **111** (1958) 1187
3) R. Karplus, C. M. Sommerfield and E. H. Wichman (preprint)

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Singularities and Discontinuities of Feynman Amplitudes*

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The Landau singularities of the amplitude calculated from an arbitrary Feynman graph are considered. It is shown that the discontinuity across a branch cut starting from any Landau singularity is obtained by replacing Feynman propagators by delta functions for those lines which appear in the Landau diagram. The general formula is a simple generalization of the unitarity condition. The discontinuity is then considered as an analytic function of the momenta and masses; it is shown that its singularities are a subclass of the singularities of the original amplitude which corresponds to Landau diagrams with additional lines. The general results are illustrated by application to some single loop graphs. In particular, the general formula gives an immediate calculation of the Mandelstam spectral function for fourth-order scattering. Singularities not of the Landau type are discussed and illustrated by the third-order vertex part.

I. INTRODUCTION

KARPLUS, Sommerfield, and Wichman[1] and Landau[2] have emphasized the importance of examining the analyticity of the amplitudes corresponding to Feynman graphs, and have discussed some simple graphs in detail. Landau has also given a criterion for determining the position of certain singularities of the amplitude for an arbitrary graph. In this paper we shall derive a formula for the discontinuity across a cut starting from any one of Landau's branch points, and shall determine where this discontinuity is singular. The result is a very natural generalization of the well-known expression, given by the unitarity condition, for the discontinuity across a cut starting from any physical threshold. The general result is extremely useful for analyzing spectral representations. For example, it leads immediately to an explicit expression for the Mandelstam spectral function for the fourth-order scattering amplitude.[3]

Before proceeding with the calculation, let us recapitulate Landau's discussion. He considers the amplitude

$$F=\int B\prod(d^4k)A_1^{-1}\cdots A_N^{-1} \tag{1}$$

(where $A_i=M_i^2-q_i^2$ and B is an arbitrary polynomial) corresponding to a graph with N internal lines and n independent loops. In (1) and the following we adhere closely to Landau's notation. The q_i are linear combinations of the k_i and the external momenta p_i. On its principal branch F has no singularities for sufficiently small, real p_i^2; if the M_i^2 are positive, we may take the p_i^2 to be positive without passing a singularity, and begin the investigation with real p_{i4} and imaginary $\mathbf{p}_i$. We denote by z_a the independent invariants formed from the p_i.

Now introduce the Feynman parametrization

$$F=(N-1)!\int\prod(d\alpha)\prod(d^4k)BD^{-N}\delta(1-\bar{\alpha}), \tag{2}$$

where $D=\sum_{i=1}^N\alpha_iA_i$ and $\bar{\alpha}=\sum\alpha_i$. Let $\varphi=\max_k(D)$ (the maximization is carried out with real k_{i4} and imaginary $\mathbf{k}_i$). According to Landau, if $\min\varphi>0$, F is nonsingular, where the minimum is taken with respect to nonnegative α's satisfying $\bar{\alpha}=1$. As the p_i^2 are increased, the first singularity of F occurs when $\min\varphi\to 0$. This, Landau shows, means that for each i

$$\alpha_iA_i=0, \tag{3}$$

and for each closed loop

$$\sum\alpha_iq_i=0, \tag{4}$$

where the sum is extended over all the lines in the loop; moreover, (4) must be satisfied with nonnegative α's. Landau pointed out that a singularity exists when (4) is satisfied with arbitrary α_i, but did not give an explicit proof of this; as this point is important to our subsequent discussion we show that this follows from an analytic continuation in the internal masses, and the continuity theorem for singularity surfaces.[4]

The following remarks are contained implicitly in Landau's paper.

Let D_m be obtained from D by setting the $\alpha_i=0$ for $i>m$, and let $\varphi_m=\max_k(D_m)$. If for some α_i, $\max_k(D_m)$ occurs for $q_i^2=M_i^2(i\leqslant m)$, then for any other nonnegative $\alpha_i(i\leqslant m)$, $\varphi_m\geqslant 0$. Now, we may choose the M_i^2 for $i>m$ so large that φ_m is the minimum of φ for nonnegative α's. For any $\alpha_i>0(i\leqslant m)$ and $p_i^2>0$ we determine q_i which satisfy (4) (this is just the maximization problem) and *define* for $i\leqslant m$ masses M_i by the equation $q_i^2=M_i^2$. Hence masses exist such that any

* Supported in part by the U. S. Atomic Energy Commission and by the Alfred P. Sloan Foundation, Inc.
† Alfred P. Sloan Foundation Fellow.
[1] R. Karplus, C. M. Sommerfield, and E. H. Wichman, Phys. Rev. 111, 1187 (1958); 114, 375 (1959).
[2] L. D. Landau, Nuclear Phys. 13, 181 (1959). *Note added in proof.* Results similar to Landau's were also obtained by J. C. Taylor [Phys. Rev. 117, 261 (1960)], which the author received after submission of this paper.
[3] S. Mandelstam, Phys. Rev. 115, 1742 (1959).
[4] H. Behnke and P. Thullen, *Theorie der Functionen Mehrerer Komplexer Veränderlichen* (Springer-Verlag, Berlin, 1934), p. 49.

"Landau diagram" corresponds to the first singularity. There are two cases to be considered, which can easily be distinguished upon inspection of the Landau diagram. If some of the masses obtained by the procedure described are constant, or satisfy a relation independent of the α's and the z's, then we do not in general have either a solution to (4) or a singularity. Otherwise, as the α's and z's are varied these masses take on all possible values, in which case it follows from the continuity theorem that for any internal masses there is always a singularity when Eqs. (3) and (4) are satisfied, although this singularity might not appear on the principal sheet of the Riemann surface.

In order to discuss the analytic continuations of F, we eliminate the delta function from (2), by replacing the α_i by $\lambda\alpha_i$, multiplying by a suitable entire function of λ (say $e^{-\lambda}$) and integrating over λ. This gives the equation

$$F=(N-1)!\int\prod(d\alpha)\prod(d^4k)BD^{-N}\tilde{\alpha}^{-1}\exp(-\tilde{\alpha}^{-1}). \quad (5)$$

In (5), the α_i vary independently over any suitable contours from 0 to ∞. We may use this equation to interpret Landau's conditions in the complex region. We use an idea introduced by Hadamard,[5] which has already been exploited in a similar problem by Eden.[6] If we first integrate over the $k_{i\mu}$, we obtain an integrand which is singular when φ vanishes, where in the general case φ is an extremum of D. The singularities of F occur when some of the α_i are fixed at the lower limit of integration, while the contours over which the remaining α_i are integrated are trapped between coalescing singularities. In other words, φ must have a double zero with respect to each of the free variables, which leads directly to Landau's conditions (3) and (4). It is also necessary that for z_a in the neighborhood of a singularity of F, the contours actually pass between these nearly coalescent zeros. We know that this occurs when we consider the first singularity; we obtain an illustration of the continuity theorem if we note that when the M_i are varied, if the z_a are simultaneously varied so as to keep the zeros in a nearly coalescent configuration, the contours must remain entrapped.

Since the integrand in (5) is always singular when $\tilde{\alpha}=0$, if D vanishes for $\tilde{\alpha}=0$ the condition of a double zero with respect to the free variables is relaxed. In this case we might have a singularity even if conditions (3) and (4) do not hold, although such a singularity could never appear on the principal sheet. We shall show, in Sec. III, that an "anomalous" singularity of this type actually occurs in the third-order vertex.

[5] J. Hadamard, Acta Math. **22**, 55 (1898).

[6] R. J. Eden, Proc. Roy. Soc. (London) **A210**, 388 (1952). *Note added in proof*. After submission of this paper, the author received two papers containing a similar discussion of the complex singularities: J. C. Polkinghorne and G. R. Screaton, Nuovo cimento **15**, 289 (1960); J. Tarski, J. Math. Phys. **1**, 154 (1960).

II. DISCONTINUITIES OF FEYNMAN AMPLITUDES

A. Calculation of the Discontinuity

We shall prove the following theorem: Let F denote the amplitude defined by Eq. (1), and let F_m denote the discontinuity of F across a branch cut starting from a singularity defined by Landau's conditions (3) and (4) in which $A_i=0$ for $i\leqslant m$; then

$$F_m=(2\pi i)^m \times\int\frac{B\prod(d^4k)\delta_p(q_1^2-M_1^2)\cdots\delta_p(q_m^2-M_m^2)}{A_{m+1}\cdots A_N}. \quad (6)$$

(The notation implies a particular ordering of the lines.) The subscript p on the delta functions means that only the contribution of the "proper" root of $q_i^2=M_i^2$ is to be taken. Equation (6) is a simple generalization of well-known results, and follows directly from the Hadamard-Eden analysis.

Consider the contracted Feynman graph obtained by fusing the vertices connected by the lines $i>m$. Let ν be the number of independent loops in this contracted graph. We can choose the k_j so that the $q_i(i\leqslant m)$ depend only on those k_j for which $j\leqslant\nu$. If the $m\times 4\nu$ matrix

$$J_{i,j\mu}=\partial q_i^2/\partial k_{j\mu}$$

is of rank m, we may choose as integration variables $\xi_i=q_i^2$ for $i\leqslant m$, and $4\nu-m$ additional variables. The q_i^2 are the squared distances between certain points in momentum space, and the ξ_i for $m<i\leqslant 4\nu$ may be interpreted as related angle variables. We shall discuss later the circumstance that $J_{i,j\mu}$ has a rank smaller than m for all $k_{j\mu}$. If the rank is too small only when the $k_{j\mu}$ satisfy particular relations, these exceptional points may in general be avoided by appropriate indentations of the $k_{j\mu}$ contours. We therefore obtain

$$F=\int_{a_1}^{b_1}dq_1^2\cdots\int_{a_m}^{b_m}dq_m^2 \times\int\frac{\prod_{m<i\leqslant 4\nu}(d\xi_i)\prod_{j>\nu}(d^4k_j)}{JA_1\cdots A_N}, \quad (7)$$

where

$$J=\det(\partial\xi_i/\partial k_{j\mu}).$$

The limits of integration (a_j,b_j) for the q_j^2 integration are the extrema of q_j^2 for fixed $q_i^2(i<j)$. This leads to the equations (for each loop of the contracted graph)

$$\sum_{(i\leqslant j)}\beta_i q_{i\mu}=0, \quad (8)$$

where the β_i are Lagrange multipliers. From (8) for $j=m$ we see that Landau's conditions (3), (4) imply that when a singularity develops, the point where the $A_i=0$ for $i\leqslant m$ lies on the boundary of the region of integration. Equation (8) also shows that the rank of $J_{i,j\mu}$ is always too small on the boundary of the integration region, but this gives no difficulty. In certain

cases each set of the q_i^2 corresponds to two points in momentum space; in these cases we interpret the q_m^2 integration as being taken over the closed contour which encloses the two points a_m and b_m where J is singular.[7]

For brevity we denote by z a point in the (many-sheeted) space of the invariants. Let z_0 denote any point on the singularity surface in question which does not also lie on some other singularity surface.

We first suppose that all the integrations in (7) have been performed, except that over q_1^2. Then we write

$$F=\int_{a_1}^{b_1} dq_1^2(M_1^2-q_1^2)^{-1}F_{(1)}(q_1^2). \tag{9}$$

Now, by hypothesis, (A) F is singular when $z\to z_0$, and (B) F would not be singular at z_0 if the factor $(M_1^2-q_1^2)^{-1}$ were absent or if the mass M_1 were changed. Therefore, the contour of the q_1^2 integration must pass between the pole $q_1^2=M_1^2$ and a singularity of $F_{(1)}(q_1^2)$ at $q_1^2=Q^2$, where $Q^2\to M_1^2$ when $z\to z_0$. We may replace this contour by one on the other side of the pole $q_1^2-M_1^2$ and a very small circle enclosing this pole, where the contour which avoids the pole gives a contribution to F which is regular in the neighborhood of z_0. The singular part of F is therefore

$$F_s=\pm 2\pi i F_{(1)}(M_1^2). \tag{10}$$

The argument given is not sufficient to determine the sign.

After applying the foregoing argument in succession to the variables $q_2^2\cdots q_{m-1}^2$, we obtain

$$F_s=\int_{a_m}^{b_m} dq_m^2(M_m^2-q_m^2)^{-1}F_{(m)}(q_m^2). \tag{11}$$

In (11), a_m and b_m are the limits calculated with $q_1^2=M_i^2$ for $i<m$. When $z\to z_0$, it follows from (8) that one of these limits coincides with the point $q_m^2=M_m^2$. It is obvious that the discontinuity across a branch cut starting from z_0 is $2\pi i F_{(m)}(M_m^2)$. When the q_m^2 integration is taken over a contour enclosing the points a_m and b_m, the two branches of F_s are determined by whether the pole $q_m^2=M_m^2$ lies inside this contour or not, so we obtain the same result.

We now *define* the sign of F_m by analytic continuation from the case where the masses are such that the singularity in question is the first encountered as the z_a are continued through real values from the singularity-free region, and z is a real point just beyond this singularity. It was shown in the Introduction that it is possible to do this. We define the discontinuity $F_m(z)$ to be the difference between $F(z)$ as calculated by giving the masses small *negative* imaginary parts and that calculated with small *positive* imaginary parts; that is,

$$F_m(z)=F_{-i\epsilon}(z)-F_{+i\epsilon}(z). \tag{12}$$

Now consider the q_m^2 integration: Equation (12) implies that the discontinuity is given by a clockwise contour around the pole. But the same result must hold for all q_i^2. This proves Eq. (6) for the case that the rank of the matrix $\partial q_i^2/\partial k_{j\mu}$ is equal to m, except that in transforming back to the $k_{i\mu}$ we must be careful to keep only the contribution from the proper root of $q_i^2=M_i^2$.

There are two cases in which the rank of $J_{i,j\mu}$ is too small; either this happens only for z which satisfy some particular relation, which restricts these z to lie on some surface, or else it occurs identically, for all z. In the first case, (6) is valid for all nonexceptional z, but the discontinuity might be singular when $J_{i,j\mu}$ is singular. If the rank is always too small, as when $m>4\nu$, we consider the singularity obtained by eliminating a sufficient number of lines (say for $m'<i\leqslant m$) that the rank of the reduced matrix $\partial q_i^2/\partial k_{i\mu}$ is m'. The singularity of the larger matrices implies that the eliminated q_i^2 can be expressed in terms of the q_i^2 for $i\leqslant m'$. Hence when we evaluate the discontinuity $F_{m'}$ by Eq. (6), we find that $F_{m'}$ has not a branch point but a pole when one of the eliminated A_i vanishes. These exceptional cases will be illustrated in Sec. III.

B. Singularities of the Discontinuity Function

We may think of $F_m(z)$ as the difference between the values of $F(z)$ on two different sheets, so the singularity surfaces of $F_m(z)$ will be contained among those of $F(z)$. We discuss these singularities by introducing $N-m$ Feynman parameters $\alpha_i(i>m)$ and repeating Landau's calculation. When we integrate over the $k_{i\mu}$, we obtain a singularity for those values of the α_i for which

$$\varphi=\text{Extremum}_k(\textstyle\sum_{i>m}\alpha_i A_i)$$

vanishes. However, the variables $k_{i\mu}$ are not all independent, because they satisfy the constraints $A_i=0$ for $i\leqslant m$. These constraints are introduced into the extremization by using m Lagrange multipliers, which we also call $\alpha_i(i\leqslant m)$. This leads to the equation $\sum\alpha_i q_{i\mu}=0$, which is identical to (4). The integration over the Feynman parameters is singular when some of them are zero, and φ is a vanishing extremum with respect to the rest. This leads to Eq. (3) for $i>$m. We are not allowed to omit any of the conditions $A_i=0$ for $i\leqslant m$, so the singularities of $F(z)$ which are also singularities of $F_m(z)$ correspond to Landau diagrams in which lines have been added to the Landau diagram which defined the original singularity. The other singularities of F necessarily appear on both sheets and cancel when we calculate the difference. As we have pointed out before, there is also a possibility of non-Landauian singularities.

[7] For some graphs with more than one loop, several of the q_i^2 integrations need to be interpreted in this way.

Let us denote by $F_{m,m'-m}(z)$ the discontinuity of $F_m(z)$ across a branch cut starting from the branch point for which $A_i=0$ for $m<i\leqslant m'$. We calculate $F_{m,m'-m}$ by the same method used to calculate F_m; we use the q_i^2 as variables for $i\leqslant m'$. It is clear that all the steps in the proof (except for determination of the sign) are identical. Moreover, we find that

$$F_{m,m'-m}(z)\equiv F_{m'}(z). \tag{13}$$

[We use Eq. (13) to define the sign of $F_{m,m'-m}$.] It may be noted that it can be proved independently (by extending the argument in the Introduction) that the singularity of F which corresponds to $A_i=0$ for $i\leqslant m'$ only appears on one of two adjacent sheets connected by the branch point corresponding to $A_i=0$ for $i\leqslant m$.

C. Unitarity Condition

Consider two graphs, each with m outgoing lines, and with r and s incoming lines, respectively. Let F and G denote the corresponding amplitudes. The unitarity of the S matrix implies that these two graphs give a contribution to the imaginary part of the T matrix (for r outgoing and s incoming particles) which is, apart from numerical factors and with neglect of the spins of the particles,

$$\mathcal{T}_{rs(m)}=\int d\tau_m F^*G, \tag{14}$$

where $d\tau_m$ is the volume element in the phase space of m particles. Let $\mathbf{q}_i$ and W_i denote the momenta and energies of these m particles. As a consequence of momentum conservation, the $\mathbf{q}_i$ depend linearly on $m-1$ integration variables $\mathbf{k}_i$. With a covariant normalization of states, we have

$$d\tau_m=\frac{d^3k_1\cdots d^3k_{m-1}}{(2W_1)\cdots(2W_m)}\delta(\sum W_i-E), \tag{15}$$

where E is the total energy. We may introduce $m-1$ new integration variables k_{i4} and write (15) as follows:

$$d\tau_m=d^4k_1\cdots d^4k_{m-1}\delta_p(q_1^2-M_1^2)\cdots\delta_p(q_m^2-M_m^2). \tag{16}$$

In (16) the q_{i4} are the same functions of the k_{i4} as the $\mathbf{q}_i$ are of the $\mathbf{k}_i$. The subscript p means that only the "proper" root of $q_i^2=M_i^2$, that for which q_{i4} is positive, is to be considered when the integrations are carried out.

Equation (14) is first obtained for real momenta. To continue it to the complex region we introduce the explicit forms of G and F, with the notation that q_i is the momentum of any internal line, and k_i is any integration variable. Then (14) becomes

$$\mathcal{T}_{rs(m)}=\int\frac{\Pi(d^4k)B\delta_p(q_1^2-M_1^2)\cdots\delta_p(q_m^2-M_m^2)}{A_{m+1}\cdots A_N}, \tag{17}$$

where $A_i=M_i^2+i\epsilon-q_i^2$ for lines belonging to the graph F, and $A_i=M_i^2-i\epsilon-q_i^2$ for lines belonging to the graph G.

Equation (17) is just a special case of the general discontinuity formula (6) for the graph obtained by joining the graphs F and G by the m common lines. In (17) the analytic continuations have been defined in a particular way (by the $\pm i\epsilon$ rule), while in (6) the masses may be considered to be arbitrary. The discussion in Sec. II.B of the location of the singularities of $F_m(z)$ applies without modification to $\mathcal{T}_{rs(m)}$.

The correspondence between the unitarity condition (17) and the general discontinuity formula (6) suggests that the general discontinuity may be looked on as a pseudounitarity condition. The particles, instead of being divided into the two groups of "initial" and "final" particles, may be divided into three or more groups.

III. ILLUSTRATIONS

In this section we illustrate the results derived in Sec. II by applying them to the three graphs shown in Fig. 1.

A. Fourth-Order Scattering

The singularities correspond to the vanishing of the following combinations of the A_i: (13), (24), (12), (23), (34), (41), (123), (134), (124), (234), and (1234). The ordinary threshold is the (13) singularity. The corresponding discontinuity is obtained by replacing A_1^{-1} and A_3^{-1} by $2\pi i\delta_p(A_1)$ and $2\pi i\delta_p(A_3)$. The discussion in Sec. II.B shows that this discontinuity has only the singularities (13), (123), (134), and (1234). The Mandelstam spectral function[3] is, apart from a factor of four, the discontinuity of this discontinuity function across the (1234) singularity, which is

$$F_4=\int d^4k\delta_p(q_1^2-M_1^2)\cdots\delta_p(q_4^2-M_4^2). \tag{18}$$

Reverting to the variables used in the proof of (6),

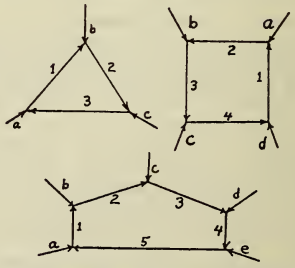

FIG. 1. Feynman graphs considered in Sec. III.

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we have

$$F=\int J^{-1}dq_1^2\cdots dq_4^2\delta(q_1^2-M_1^2)\cdots\delta(q_4^2-M_4^2) \quad (19)$$
$$=J^{-1},$$

where $J=\det\partial q_i^2/\partial k_\mu=2^4\det q_{i\mu}$ is evaluated for $q_i^2=M_i^2$. The result of Mandelstam[3] and Kibble[8] is obtained from (18) by noting that $[\det q_{i\mu}]^2=\det q_i q_j$.

The reader will recognize $\det q_{i\mu}$ as the volume of the four-dimensional parallelepiped constructed with the q_i as edges. The vectors q_i have lengths M_i, and they have such directions that when drawn from a common vertex Q, their ends are vertices of the tetrahedron constructed from the external momenta (see Fig. 2). Complex vectors are to be used in drawing the figure, when necessary. This figure (a simplex) is one corner of the parallelepiped; its volume V is 1/4! times the volume of the parallelepiped. Hence $J=2^4 4!V$.

Landau's condition for the location of the (1234) singularity is that the point Q should lie in the hyperplane of the tetrahedron. In this case $V=0$. It should

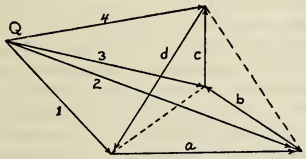

FIG. 2. The Mandelstam spectral function is the reciprocal of the volume of this figure.

be noted that the transformation from the k_μ to the q_i^2 is singular when the tetrahedron degenerates to a planar figure. But $4V$ is the product of the volume of the tetrahedron and the altitude of the point Q from the hyperplane of the tetrahedron, and when the volume of the tetrahedron vanishes, the altitude, for *fixed* lengths of the q_i, becomes infinite in such a way that V^{-1} is analytic.

B. Third-Order Vertex

The discontinuity across the (123) branch cut is

$$F_3=\frac{1}{2\pi}\int d^4k\delta_p(q_1^2-M_1^2)\delta_p(q_2^2-M_2^2)\delta_p(q_3^2-M_3^2). \quad (20)$$

Consider the point Q whose squared distances from the vertices of the triangle (p_a, p_b, p_c) are q_i^2 (see Fig. 3). The locus of Q in four-dimensional space is a circle whose radius κ is the altitude of Q from the plane of the triangle. Transforming to new variables, we have

$$d^4k=\kappa d\varphi d^3k=\kappa d\varphi\prod(dq_i^2)J_3^{-1}, \quad (21)$$

[8] T. W. B. Kibble, Phys. Rev. **117**, 1159 (1960).

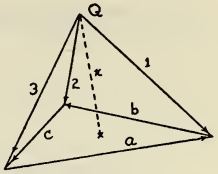

FIG. 3. Geometrical construction associated with the third-order vertex.

where $J_3=8\det q_{i\alpha}$ is a 3×3 determinant. Hence we obtain

$$F_3=\kappa J_3^{-1}. \quad (22)$$

Now $\det q_{i\alpha}$ is 3! times the volume of the tetrahedron in Fig. 3, which in turn is $\frac{1}{3}\kappa\mathfrak{A}$, where $\mathfrak{A}$ is the area of the triangle. Therefore,

$$F_3=2^{-4}\mathfrak{A}^{-1}$$
$$=\tfrac{1}{4}\{p_a^4+p_b^4+p_c^4-2p_a^2p_b^2-2p_a^2p_c^2-2p_b^2p_c^2\}^{-\frac{1}{2}}. \quad (23)$$

We see that F_3, and therefore also F on at least one sheet, is singular when $\mathfrak{A}=0$. In this example, a singularity of the matrix $\partial q_i^2/\partial k_\mu$ actually is associated with a singularity of F. The singularity can be shown to correspond, in terms of the Feynman parametrization discussed in the Introduction, to the case $\alpha_1+\alpha_2+\alpha_3=0$.

C. Example of Redundant Lines

Consider the graph shown in Fig. 1 which has five lines in one loop. Landau's procedure shows there is a singularity when all five $A_i=0$, but this is not a branch point. The discontinuity across the (1234) branch cut is shown by the method of Sec. III.A to be

$$F_4=J^{-1}(q_5^2-M_5^2)^{-1}, \quad (24)$$

where J and q_5^2 are functions of the external momenta and of $M_1,\cdots,M_4$. When the external momenta are such that $q_5^2=M_5^2$, F_4 has a pole. Since F_4 is the difference between values of the amplitude F on two adjacent sheets, and since the (12345) singularity only appears on one of them, F also has a pole. The location of the pole corresponds to the possibility of drawing the Landau diagram with four-dimensional vectors; the nonexistence of a branch cut corresponds to the impossibility of buckling the diagram into an extra dimension.

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Determination of the Pion-Nucleon Scattering Amplitude from Dispersion Relations and Unitarity. General Theory

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A method is proposed for using relativistic dispersion relations, together with unitarity, to determine the pion-nucleon scattering amplitude. The usual dispersion relations by themselves are not sufficient, and we have to assume a representation which exhibits the analytic properties of the scattering amplitude as a function of the energy and the momentum transfer. Unitarity conditions for the two reactions $\pi+N \rightarrow \pi+N$ and $N+\bar{N} \rightarrow 2\pi$ will be required, and they will be approximated by neglecting states with more than two particles. The method makes use of an iteration procedure analogous to that used by Chew and Low for the corresponding problem in the static theory. One has to introduce two coupling constants; the pion-pion coupling constant can be found by fitting the sum of the threshold scattering lengths with experiment. It is hoped that this method avoids some of the formal difficulties of the Tamm-Dancoff and Bethe-Salpeter methods and, in particular, the existence of ghost states. The assumptions introduced are justified in perturbation theory.

As an incidental result, we find the precise limits of the region for which the absorptive part of the scattering amplitude is an analytic function of the momentum transfer, and hence the boundaries of the region in which the partial-wave expansion is valid.

1. INTRODUCTION

IN recent years dispersion relations have been used to an increasing extent in pion physics for phenomenological and semiphenomenological analyses of experimental data,[1] and even for the calculation of certain quantities in terms of the pion-nucleon scattering amplitude.[2] It is therefore tempting to ask the question whether or not the dispersion relations can actually replace the more usual equations of field theory and be used to calculate all observable quantities in terms of a finite number of coupling constants—a suggestion first made by Gell-Mann.[3] At first sight, this would appear to be unreasonable, since, although it is necessary to use all the general principles of quantum field theory to derive the dispersion relations, one does not make any assumption about the form of the Hamiltonian other than that it be local and Lorentz-invariant. However, in a perturbation expansion these requirements are sufficient to specify the Hamiltonian to within a small number of coupling constants if one demands that the theory be renormalizable and therefore self-consistent. It is thus very possible that, even without a perturbation expansion, these requirements are sufficient to determine the theory. In fact, if the "absorptive part" of the scattering amplitude, which appears under the integral sign of the dispersion relations, is expressed in terms of the scattering amplitude by means of the unitarity condition, one obtains equations which are very similar to the Chew-Low[4] equations in static theory. These equations have been used by Salzman and Salzman[5] to obtain the pion-nucleon scattering phase shifts.

It is the object of this paper to find a relativistic analog of the Chew-Low-Salzman method, which could be used to calculate the pion-nucleon scattering amplitude in terms of two coupling constants only. As in the static theory, the unitarity equation will involve the transition amplitude for the production of an arbitrary number of mesons, and, in this case, of nucleon pairs as well. In order to make the equations manageable, it is necessary to neglect all but a finite number of processes; as a first approximation, the "one-meson" approximation, we shall neglect all processes except elastic scattering.

The equations obtained from the dispersion relations and the one-meson approximation differ from the static Chew-Low equations in two important respects. Whereas, in the static theory, there was only P-wave scattering, we now have an infinite number of angular momentum states, and the crossing relation, if expressed in terms of angular momentum states, would not converge. Further, in the relativistic theory, the dispersion relations involve the scattering amplitude in the "unphysical" region, i.e., through angles whose cosine is less than -1. For these reasons, the method of procedure will be more involved than in the static theory. We shall require, not only the analytic properties of the scattering amplitude as a function of energy for fixed momentum transfer, which are expressed by the dispersion relations, but its analytic properties as a function of both variables. The required analytic properties have not yet been proved to be consequences of microscopic causality. In order to carry out the proof,

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[1] Chew, Goldberger, Low, and Nambu, Phys. Rev. **106**, 1345 (1957). This paper contains further references.

[2] Chew, Karplus, Gasiorowicz, and Zachariasen, Phys. Rev. **110**, 265 (1958).

[3] M. Gell-Mann, *Proceedings of the Sixth Annual Rochester Conference High-Energy Physics, 1956* (Interscience Publishers, Inc., New York, 1956), Sec. III, p. 30.

[4] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

[5] G. Salzman and F. Salzman, Phys. Rev. **108**, 1619 (1957).

one would almost certainly have to consider simultaneously several Green's functions together with the equations connecting them which follow from unitarity. It is unlikely that such a program will be carried through in the immediate future. However, if the solution obtained by the use of these analytic properties were to be expanded in a perturbation series, we would obtain precisely those terms of the usual perturbation series included in the one-meson approximation. The assumed analytic properties are, therefore, probably correct, at any rate in the one-meson approximation.

As we have to resort to perturbation theory in order to justify our assumptions, we do not yet have a theory in which the general principles of quantum theory are supplemented only by the assumption of microscopic causality. Nevertheless, the approximation scheme used has several advantages over the approximations previously applied to this problem, such as the Tamm-Dancoff or Bethe-Salpeter approximations. It refers throughout only to renormalized masses and coupling constants. The Tamm-Dancoff equations, by contrast, are unrenormalizable in higher approximations and the Bethe-Salpeter equations, while they are covariant and therefore renormalizable in all approximations, present difficulties of principle when one attempts to solve them. Further, we may hope that the one-meson approximation is more accurate than the Tamm-Dancoff approximations. The latter assumes that those components of the state vector containing more than a certain number of bare mesons are negligibly small—an approximation that is known to be completely false for the experimental value of the coupling constant. The one-meson approximation, on the other hand, assumes that the cross section for the production of one or more *real* mesons is small except at high energies. While this approximation is certainly not quantitatively correct, it is nevertheless probably a good deal more accurate than the Tamm-Dancoff approximation. Finally, the one-meson approximation, unlike the Tamm-Dancoff or Bethe-Salpeter approximations, possesses crossing symmetry. Now it is very probable that the "ghost states" which have been plaguing previous solutions of the field equations are due to the neglect of crossing symmetry. As evidence of this, we may cite the case of charged scalar theory without recoil, for which the one-meson approximation has been solved completely.[6,7] The solution obtained with neglect of the crossing term possesses the usual ghost state if the source radius is sufficiently small. The Lee model,[8] which has no crossing symmetry, shows a similar behavior. If the crossing term in the charged scalar model is included, however, there is no ghost state.

It has been pointed out by Castillejo, Dalitz, and Dyson[7] that the dispersion relations, at any rate in the charged scalar model, do not possess a unique solution. This might have been expected, since it is possible to alter the Hamiltonian without changing the dispersion relations. One simply has to introduce into the theory a baryon whose mass is greater than the sum of the masses of the meson and nucleon. Such a baryon would be unstable, and would therefore not appear as a separate particle or contribute a term to the dispersion relations. In perturbation theory, the simplest of the solutions found by Castillejo, Dalitz, and Dyson, i.e., the solution without any zero in the scattering amplitude, agrees with the solution obtained from a Hamiltonian in which there are no unstable particles, and the more complicated solutions correspond to the existence of unstable baryons. We shall assume that this is so independently of perturbation theory, and shall concern ourselves with the simplest solution. There is no physical reason why one of the other solutions may not be the correct one, but it seems worthwhile to try to compare with experiment the consequences of a theory without unstable particles. It should in any case be emphasized that the ambiguity is not a specific feature of this method of solution, but is inherent in the theory itself. The difference is that, in other methods, it occurs in writing down the equations, whereas in this method it occurs in solving them.

In Sec. 2 we shall discuss the analytic properties of the scattering amplitude, and, in Sec. 3, we shall show how these properties can be used together with the unitarity condition to solve the problem. We shall in this section ignore the "subtraction terms" in the dispersion relations. As in the corresponding static problem, we have to use an iteration procedure in which the crossing term is taken from the result of the previous iteration. The details of this solution will be entirely different from the static problem, the reason being that the part of the amplitude corresponding to the lowest angular momentum states, which is a polynomial in the momentum transfer, actually appears as a subtraction term in the dispersion relation with respect to this variable and has thus not yet been taken into account. In this and the next section we shall also be able to specify details of the analytic representation that were left undetermined in Sec. 2, in particular, we shall be able to give precise limits to the values of the momentum transfer within which the partial-wave expansion converges. In Sec. 4 we shall investigate the subtraction terms in the dispersion relations. We shall find that, in order to determine them, we shall require the unitarity condition for the lowest angular momentum states, not only in pion-nucleon scattering, but also in the pair-annihilation reaction $N+\bar{N} \rightarrow 2\pi$, which is represented by the same Green's function. The coupling constant for meson-meson scattering is thus introduced into the theory; as its value is not known experimentally it will have to be determined by fitting one of the results of the calculation, such as the sum of the S-wave scattering lengths at threshold, with experiment. The calculations

[6] T. D. Lee and R. Serber (unpublished).
[7] Castillejo, Dalitz, and Dyson, Phys. Rev. 101, 453 (1956).
[8] K. W. Ford, Phys. Rev. 105, 320 (1957).

of these low angular momentum states would be done in the same spirit as the Chew-Low calculations, and the details will not be given in this paper. We thus have a procedure in which the first few angular momentum states are calculated by methods similar to those used in the static theory, while the remaining part of the scattering amplitude, which will be called the "residual part," is calculated by a different procedure which does not make use of a partial-wave expansion. Needless to say, the two parts of the calculation become intermingled by the iteration procedure.

It is only in the calculation of the subtraction terms that use has to be made of the unitarity condition for the pair-annihilation reaction. For the residual part, it is only necessary to use the unitarity condition for pion-nucleon scattering. Had it been possible to use the unitarity condition exactly instead of in the one-meson approximation, the result would also satisfy the unitarity condition for the annihilation reaction in a consistent theory. As it is, we find that the residual part consists of a number of terms which correspond to various intermediate states in the annihilation reaction. In Sec. 5 it is pointed out that the calculation is greatly simplified if we keep only those terms of the residual part corresponding to pair annihilation through states with fewer than a certain number of particles. Such an approximation has already been made in calculating the subtraction terms. The unitarity condition for pion-nucleon scattering is no longer satisfied except for the low angular momentum states. However, the terms neglected are of the order of magnitude of, and probably less than, terms already neglected. The two reactions of pion-nucleon scattering and pair annihilation are now treated on an equivalent footing.

It will be found that the unitarity condition, in the one-meson approximation, cannot be satisfied at all energies if crossing symmetry and the analytic properties are to be maintained. The reason is that the unitarity condition for the scattering reaction is not completely independent of the unitarity condition for the "crossed" reaction with the two pions interchanged, and they contradict one another if an approximation is made. There is, of course, no difficulty in the region where the one-meson approximation is exact. For sufficiently small values of the coupling constant, we shall still be able to obtain a unique procedure. For values of the coupling constant actually encountered, one part of the crossing term may have to be cut off at the threshold for pair production in pion-nucleon scattering. It is unlikely that the result will be sensitive to the form and the precise value of the cutoff.

2. DISPERSION RELATIONS AND ANALYTICITY PROPERTIES OF THE TRANSITION AMPLITUDE

The kinematical notation to be used in writing down the dispersion relations will be similar to that of Chew *et al.*[1] The momenta of the incoming and outgoing pions will be denoted by q_1 and q_2, those of the incoming and outgoing nucleons by p_1 and p_2. We can then define two invariant scalars

$$\nu=-(p_1+p_2)(q_1+q_2)/4M, \tag{2.1}$$

$$t=-(q_1-q_2)^2. \tag{2.2}$$

The latter is minus the square of the invariant momentum transfer. The laboratory energy will be given by the equation

$$\omega=\nu-(t/4M). \tag{2.3a}$$

It is more convenient to use, instead of the laboratory energy, the square of the center-of-mass energy (including both rest-masses), which is linearly related to it by the equation

$$s=M^2+\mu^2+2M\omega. \tag{2.3b}$$

The Green's function relevant to the process under consideration,

$$\pi_1+N_1\rightarrow\pi_2+N_2, \tag{I}$$

also gives the processes

$$\pi_2+N_1\rightarrow\pi_1+N_2 \tag{II}$$

and

$$N_1+\bar{N}_2\rightarrow\pi_1+\pi_2. \tag{III}$$

The matrix elements for the process II can be obtained from those for the process I by crossing symmetry; the laboratory energy and the square of the center-of-mass energy will now be

$$\omega_c=-\nu-(t/4M)=-\omega-(t/2M), \tag{2.4a}$$

$$s_c=M^2+\mu^2+2M\omega_c=2M^2+2\mu^2-s-t. \tag{2.4b}$$

The square of the momentum transfer will be $-t$ as before. For the process III, the square of the center-of-mass energy will be t. The square of the momentum transfer between the nucleon N_1 and the pion π_2 will be s_c and that between the nucleon N_1 and the pion π_1 will be s.

The kinematics for the three reactions are represented diagrammatically in Fig. 1 in which t has been plotted against ν. AB represents the line $s=(M+\mu)^2$, or $\omega=\mu$, and lines for which s is constant will be parallel to it. The region for which the process I is energetically possible is therefore that to the right of AB. However, only the shaded part of this area is the "physical region"; in the unshaded part, though the energy of the meson is greater than its rest-mass, the cosine of the scattering angle is not between -1 and $+1$. The physical region is bounded above by the line $t=0$, i.e., the line of forward scattering, and below by the line of backward scattering. Similarly CD is the line $s_c=(M+\mu)^2$; the region for which the process II is energetically possible is that to the left of CD, and the shaded area represents the physical region for this

reaction. Lines of constant energy for the reaction III are horizontal lines. The reaction will be energetically possible above the line EF, at which $t=4M^2$, and again the shaded area represents the physical region.

We now examine the analytic properties of the scattering amplitude. To simplify the writing, we shall first neglect spin and isotopic spin; the transition amplitude will then be a scalar function $A(\nu,t)$ of the two invariants ν and t. Its analytic properties as a function of ν, with t constant, are exhibited by the usual dispersion relations

$$A(\nu,t)=\frac{g^2}{2M}\left(\frac{1}{\nu_B-\nu}+\frac{1}{\nu_B+\nu}\right)+\frac{1}{\pi}\int_{\mu+(t/4M)}^{\infty}d\nu'\,\frac{A_1(\nu',t)}{\nu'-\nu}$$
$$-\frac{1}{\pi}\int_{-\infty}^{-\mu-(t/4M)}d\nu'\,\frac{A_2(\nu',t)}{\nu'-\nu},\quad(2.5)$$

where $\nu_B=-(\mu^2/2M)+(t/4M)$. In this and all subsequent such equations, the energy denominators are taken to have a small imaginary part. A_1 and A_2 are the "absorptive parts" associated with the reactions I and II, respectively, and are given by the equations

$$(2\pi)^4A_1(\nu,t)\delta(p_1+q_1-p_2-q_2)=(2\pi)^6\left(\frac{4p_{01}p_{02}q_{01}q_{02}}{M^2}\right)^{\frac{1}{2}}$$
$$\times\sum_n\langle N(p_1)\pi(q_1)|n\rangle\langle n|N(p_2)\pi(q_2)\rangle,\quad(2.6)$$

$$(2\pi)^4A_2(\nu,t)\delta(p_1+q_1-p_2-q_2)=(2\pi)^6\left(\frac{4p_{01}p_{02}q_{01}q_{02}}{M^2}\right)^{\frac{1}{2}}$$
$$\times\sum_n\langle N(p_1)\pi(-q_2)|n\rangle\langle n|N(p_2)\pi(-q_1)\rangle.\quad(2.7)$$

The symbol $\langle N(p_1)\pi(q_1)|$ denotes a state with an ingoing nucleon of momentum p_1 and an ingoing pion of momentum q_1. The sum $\sum_n$ is to be taken over all intermediate states. A_1 and A_2 are nonzero to the right of AB, and to the left of CD, respectively.

Equation (2.5) indicates that A is an analytic function of ν in the complex plane, with poles at $\pm\nu_B$, and cuts along the real axis from $\mu+(t/4M)$ to ∞ and from $-\infty$ to $-\mu-(t/4M)$.

On Fig. 1, (2.5) will be represented by an integration along a horizontal line below the ν axis. The poles will occur where this line crosses the dashed lines; apart from them, the integrand will be zero between AB and CD. Except for forward scattering, the region where the integrand is nonzero will lie partly in the unphysical region, where the energy is above threshold but the angle imaginary.

Equation (2.5) is only true as it stands if the functions A, A_1, and A_2 tend to zero sufficiently rapidly as ν tends to infinity; otherwise it will be necessary to perform one or more subtractions in the usual way. Whenever such a dispersion relation is written down, the possibility of having to perform subtractions is implied.

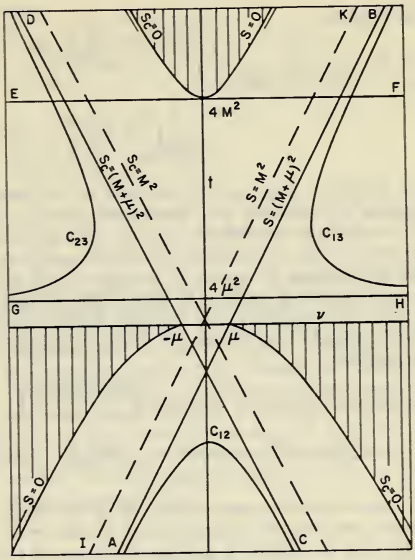

FIG. 1. Kinematics of the reactions I, II, and III.

We next wish to obtain analytic properties of A as a function of t. In order to do this we shall write the scattering amplitude, not as the expectation value of the time-ordered product of the two meson current operators between two one-nucleon states, as is done in the proof of the usual dispersion relations,[9,10] but as the expectation value of the product of a meson current operator and a nucleon current operator between a nucleon state and a meson state. Thus

$$(2\pi)^4A\delta(p_1+p_2-q_1-q_2)=(2\pi)^3\left(\frac{2p_{01}q_{02}}{M}\right)^{\frac{1}{2}}i\int dxdx'$$
$$\times e^{-iq_1x+ip_2x'}\langle N(p_1)|T\{j(x)\bar{a}(x')\}|\pi(q_2)\rangle,\quad(2.8)$$

where $a(x')$ is a nucleon current operator. From this expression, we can obtain dispersion relations in which the momentum transfer between the incoming nucleon and the outgoing pion, rather than between the two nucleons, is kept constant—the proof is exactly the same as the usual heuristic proof of the ordinary dispersion relations.[9,10] As this momentum transfer is just s_c, we obtain dispersion relations in which s_c is kept constant; if A is written as a function of s_c and t, they

[9] M. L. Goldberger, Phys. Rev. 99, 979 (1955).
[10] R. H. Capps and G. Takeda, Phys. Rev. 106, 1337 (1956).

take the form

$$A(s_c,t)=\frac{g^2}{s_c+t-M^2-2\mu^2}-\frac{1}{\pi}\int_{\infty}^{(M-\mu)^2-s_c}dt'\,\frac{A_1(s_c,t')}{t'-t}$$
$$+\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'\,\frac{A_3(s_c,t')}{t'-t}. \quad (2.9)$$

The absorptive parts in the integrand are as usual obtained by replacing the time-ordered product in (2.8) by half the commutator. The first term, in which the operators are in the order $j(x)\bar{a}(x')$, is exactly A_1, and will therefore be nonzero to the right of AB and have a δ function along IK. The second term, however, in which the operators are in the order $\bar{a}(x')j(x)$, will now be related to the process III. It will be given by the equation

$$(2\pi)^4A_3(s_c,t)\delta(p_1+q_1-p_2-q_2)=(2\pi)^6\left(\frac{4p_{01}p_{02}q_{01}q_{02}}{M^2}\right)^{\frac{1}{2}}$$
$$\times\sum_n\langle N(p_1)\bar{N}(-p_2)|n\rangle\langle n|\pi(-q_1)\pi(q_2)\rangle. \quad (2.10)$$

The state n of lowest energy will now be the two-meson state. A_3 will therefore be nonzero above the line $t=4\mu^2$, represented by GH in Fig. 1 (since t is square of the center-of-mass energy of the process III). The dispersion relation (2.10) is represented by an integration along a line parallel to CD and to the right of the line $s_c=0$. It implies that A is an analytic function of t for fixed s_c, with a pole at $t=M^2+2\mu^2-s_c$, and cuts along the real axis from $-\infty$ to $(M-\mu)^2-s_c$ and from $4\mu^2$ to ∞.

As in the usual dispersion relation, part of the range of integration in Eq. (2.9) will lie in the unphysical region. This region now includes, besides imaginary angles at permissible energies, the entire area between the lines $t=4\mu^2$ and $t=4M^2$, where there are contributions to A_3 from intermediate states with two or more pions. The rigorous proof of (2.9) is therefore much more difficult than that of (2.5), and probably cannot be carried out without introducing the unitarity equations.

By interchanging the two pions in the expression (2.8), we can obtain a third dispersion relation in which s is kept constant:

$$A(s,t)=\frac{g^2}{s+t-M^2-2\mu^2}-\frac{1}{\pi}\int_{-\infty}^{(M-\mu)^2-s}dt'\,\frac{A_2(s,t')}{t'-t}$$
$$+\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'\,\frac{A_3(s,t')}{t'-t}. \quad (2.11)$$

On Fig. 1, this would be represented by an integration along a line parallel to AB, and to the left of the line $s=0$.

Let us now try to obtain the analytic properties of A considered as a function of two complex variables. The simplest assumption we could make is that it is analytic in the entire space of the two variables except for cuts along certain hyperplanes. We can then determine the location of the cuts from the requirement that A must satisfy the dispersion relations (2.5), (2.9), and (2.11); there will be a cut when s is real and greater than $(M+\mu)^2$, a cut when s_c is real and greater than $(M+\mu)^2$, and a cut when t is real and greater than $4\mu^2$. The discontinuities across these cuts will be, respectively, $2A_1$, $2A_2$, and $2A_3$. In addition, A will have poles when $s=M^2$ and when $s_c=M^2$. By a double application of Cauchy's theorem, it can be shown that a function with cuts and poles in these positions can be represented in the form

$$A=\frac{g^2}{M^2-s}+\frac{g^2}{M^2-s_c}+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}ds'\int_{4\mu^2}^{\infty}dt'\,\frac{A_{13}(s',t')}{(s'-s)(t'-t)}$$
$$+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}ds_c'\int_{4\mu^2}^{\infty}dt'\,\frac{A_{23}(s_c',t')}{(s_c'-s_c)(t'-t)}$$
$$+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}ds'\int_{(M+\mu)^2}^{\infty}ds_c'\,\frac{A_{12}(s',s_c')}{(s'-s)(s_c'-s_c)}. \quad (2.12)$$

This is a generalization of a representation first suggested by Nambu.[11] While we have for convenience used the three variables s, s_c, and t, which are the energies of the three processes, they are connected by the relation

$$s+s_c+t=2(M^2+\mu^2), \quad (2.13)$$

so that A is really a function of two variables only. A_{13}, A_{23} and A_{12}, which will be referred to as the "spectral functions," are nonzero in the regions indicated at the top right, top left and bottom of Fig. 1. The precise boundaries C_{13}, C_{23}, and C_{12} of the regions will be determined by unitarity in the following sections; from the reasoning given up till now, all that can be said is that the regions must lie within the respective triangles as indicated, and that the boundary must approach the sides of the triangles asymptotically (or it could touch them at some finite point). The spectral functions are always zero in the physical region.

As in the case of ordinary dispersion relations, the representation (2.12) will not be true as it stands, but will require subtractions. The subtractions will modify one or both of the energy denominators in the usual way and, in addition, they will require the addition of extra terms. These terms will not now be constants, but functions of one of the variables, e.g., if there is a subtraction in the s integration of the first term, the extra term will be a function of t. These functions must then have the necessary analytic properties in their

[11] Y. Nambu, Phys. Rev. 100, 394 (1955).

variables, so that they will have the form

$$\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds' \frac{f_1(s')}{s'-s}+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds_c' \frac{f_2(s_c')}{s_c'-s_c} +\frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{f_3(t')}{t'-t}. \quad (2.14)$$

If more than one subtraction is involved, we may have similar terms multiplied by polynomials. Even if the spectral functions in (2.12) tend to zero as one of the variables tends to infinity, so that no subtraction in that variable is necessary, it is still not precluded that the corresponding term in (2.14) does not appear, as the function still has the required analytic properties. For pion-nucleon scattering, however, there is no undetermined over-all term, independent of both variables, to be added, as the requirement that the scattering amplitude for each angular momentum wave have the form $e^{i\delta}\sin\delta/k$, with $\mathrm{Im}\delta<0$, forces A to tend to zero in the physical region when both s and t become infinite.

The Nambu representations for the complete Green's functions are known to be invalid, even in the lowest nontrivial order of perturbation theory. The representation quoted here, however, restricts itself to the mass shells of the particles, and has not been shown to be invalid. In fact, in the case of Compton scattering, the fourth-order terms, which have been worked out by Brown and Feynman,[12] are found to have this representation, and, as we have stated in the introduction, all the perturbation terms included in the one-meson approximation can be similarly represented.

The dispersion relations are an immediate consequence of the representation (2.12). To obtain the usual dispersion relation (2.5), the third integral in (2.12) must be written as[13]

$$-\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty} ds' \int_{-\infty}^{t_2(s')} dt' \frac{A_{12}(s',t')}{(s'-s)(t'-t)} -\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty} ds' \int_{-\infty}^{t_2(s_c')} dt' \frac{A_{12}(s_c',t')}{(s_c'-s_c)(t'-t)}.$$

It then follows that

$$A=\frac{g^2}{M^2-s}+\frac{g^2}{M^2-s_c}+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty} ds' \frac{A_1(s',t)}{s'-s} +\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty} ds_c' \frac{A_2(s_c',t)}{s_c'-s_c}, \quad (2.15)$$

where

$$A_1(s,t)=\frac{1}{\pi}\int_{t_1(s)}^{\infty} dt' \frac{A_{13}(s,t')}{t'-t} -\frac{1}{\pi}\int_{-\infty}^{t_2(s)} dt' \frac{A_{12}(s,t')}{t'-t}, \quad (2.16)$$

$$A_2(s_c,t)=\frac{1}{\pi}\int_{t_1(s_c)}^{\infty} dt' \frac{A_{23}(s_c,t')}{t'-t} -\frac{1}{\pi}\int_{-\infty}^{t_2(s_c)} dt' \frac{A_{12}(s_c,t')}{t'-t}. \quad (2.17)$$

Equation (2.15) is, however, just the dispersion relation (2.5), since s, s_c, and ν are connected by the relations (2.4) and t is being kept constant. We also see that the absorptive parts A_1 and A_2 themselves satisfy dispersion relations in t, with s (or s_c) constant; the imaginary parts which appear in the integrand are now simply the spectral functions. Equation (2.16) will be represented in Fig. 1, by an integration along a line parallel to AB and to the right of it. The limits t_1 and t_2 are the points at which this line crosses the curves C_{13} and C_{12}. They satisfy the inequalities

$$t_1>4\mu^2, \quad (2.18a)$$

$$t_2<(M-\mu)^2-s. \quad (2.18b)$$

A_1 will be nonzero for $s>(M+\mu)^2$, as it should, as long as the curves C_{13} and C_{12} approach the line AB at some point and do not cross it.

The dispersion relations (2.9) and (2.11) can be proved from (2.12) in a similar way; the absorptive part A_3 will then satisfy a dispersion relation in ν with s constant:

$$A_3=\frac{1}{\pi}\int_{\nu_3(t)}^{\infty} d\nu' \frac{A_{13}(\nu',t)}{\nu'-\nu}-\frac{1}{\pi}\int_{-\infty}^{-\nu_3(t)} d\nu' \frac{A_{23}(\nu',t)}{\nu'-\nu} \quad (2.19)$$

This dispersion relation will be represented by an integration along a horizontal line above GH. ν_3 and $-\nu_3$ will be the points at which the line of integration crosses C_{13} and C_{23}.

Finally, then, the scattering amplitude A satisfies dispersion relations in which any of the quantities t, s_c, and s are kept constant. Further, it follows from (2.12), by the reasoning just given, that the values of the quantity which is being kept constant need no longer be restricted in sign. Thus, for example, we now know the analytic properties of A, as a function of momentum transfer, for fixed energy greater than (as well as less than) $(M+\mu)^2$. They are given by the dispersion relation (2.11), so that A is an analytic function of the square of the momentum transfer, with a pole at $t=M^2+2\mu^2-s$, and cuts along the real axis from $t=4\mu^2$ to ∞ and from $t=-\infty$ to $(M-\mu)^2-s$. For $s>(M+\mu)^2$, these cuts and poles are entirely in the nonphysical region. It has already been shown rigorously

[12] L. M. Brown and R. P. Feynman, Phys. Rev. 85, 231 (1952).

[13] When we make a change of variables, we imply of course that the spectral functions still have the same value at the same point, and not that we must take the same function of the new variables.

by Lehmann[14] that A is analytic in t in an area including the physical region. The absorptive parts A_1, A_2 and A_3 will themselves satisfy dispersion relations, provided that the correct variable be kept constant (s, s_c, and t for A_1, A_2, and A_3, respectively). The weight functions for these dispersion relations are entirely in the nonphysical region, and the boundaries of the areas in which they are nonzero are yet to be determined. In particular, we see that the absorptive part A_1 has the same analytic properties as a function of the momentum transfer [for s constant and greater than $(M+\mu)^2$] as the scattering amplitude, except that there is now no pole, and the cuts only extend from t_1 to ∞ and from $-\infty$ to t_2. According to the inequalities (2.14), these cuts do not reach as far inward as the cuts of A considered as a function of the momentum transfer. This agrees with another result of Lehmann[14] who showed that the region of analyticity of A_1 as a function of t was larger than the region of analyticity of A as a function of t.

The modifications introduced into the theory by spin and isotopic spin are trivial. The transition amplitude will now be given by the expression

$$-A+\tfrac{1}{2}i\gamma(q_1+q_2)B, \tag{2.20}$$

and both A and B will have representations of the form (2.12). There will, further, be two amplitudes corresponding to isotopic spins of $\frac{1}{2}$ and $\frac{3}{2}$. It is sometimes more convenient to use the combinations

$$A^{(+)}=\tfrac{1}{3}(A^{(\frac{1}{2})}+2A^{(\frac{3}{2})}), \tag{2.21a}$$

$$A^{(-)}=\tfrac{1}{3}(A^{(\frac{1}{2})}-A^{(\frac{3}{2})}), \tag{2.21b}$$

and similar combinations $B^{(+)}$ and $B^{(-)}$. We then have the simple crossing relations

$$A^{(\pm)}(\nu,t)=\pm A^{(\pm)}(-\nu,t), \tag{2.22a}$$

$$B^{(\pm)}(\nu,t)=\mp B^{(\pm)}(-\nu,t), \tag{2.22b}$$

or, in terms of the spectral functions,

$$A_{13}{}^{(\pm)}(s,t)=\pm A_{23}{}^{(\pm)}(s_c,t), \tag{2.23a}$$

$$A_{12}{}^{(\pm)}(s,s_c)=\pm A_{12}{}^{(\pm)}(s_c,s), \tag{2.23b}$$

$$B_{13}{}^{(\pm)}(s,t)=\mp B_{23}{}^{(\pm)}(s_c,t), \tag{2.23c}$$

$$B_{12}{}^{(\pm)}(s,s_c)=\mp B_{12}{}^{(\pm)}(s_c,s). \tag{2.23d}$$

The poles in (2.12) and in the dispersion relations will only occur in the representation for $B^{(\pm)}$ (in pseudoscalar theory), and the second term will have a minus or plus sign in the equations for $B^{(+)}$ and $B^{(-)}$, respectively.

3. COMBINATION OF THE DISPERSION RELATIONS WITH THE UNITARITY CONDITION

The dispersion relations given in the previous section must now be combined with the unitarity equations in order to determine the scattering amplitude. We shall again begin by neglecting spin and isotopic spin; the unitarity condition (2.7) then becomes, in the one-meson approximation,

$$A_1(s,\cos\theta_1)=\frac{1}{32\pi^2}\frac{q}{W}\int \sin\theta_2 d\theta_2 d\phi_2\, A^*(s,\cos\theta_2) \times A(s,\cos(\theta_1,\theta_2)),$$

or

$$A_1(s,z_1)=\frac{1}{32\pi^2}\frac{q}{W}\int_{-1}^{1}dz_2\int_0^{2\pi}d\phi\, A^*(s,z_2) \times A\{s,z_1z_2+(1-z_1^2)^{\frac{1}{2}}(1-z_2^2)^{\frac{1}{2}}\cos\phi\}, \tag{3.1}$$

where $z=\cos\theta$ and $\theta_i (i=1,2)$ is a unit vector in the (θ_i,ϕ_i) direction. W is the center-of-mass energy (equal to $\sqrt{s}$), and q is the momentum in the center-of-mass system, given by the equation

$$q^2=\{s-(M+\mu)^2\}\{s-(M-\mu)^2\}/4s. \tag{3.2}$$

z is related to the momentum transfer by the simple relation

$$z=1+(t/2q^2). \tag{3.3}$$

The unitarity requirements only prove that Eq. (3.2) is true in the physical region. A_1 must then be obtained in the unphysical region by analytic continuation. In order to do this, A can be expressed as an analytic function of t or, equivalently, of z, by means of Eq. (2.11), in which the energy is kept fixed. Equation (3.3) shows that we can simply replace t by z in (2.12), so that we may write

$$A^*(s,z_2)=\frac{1}{\pi}\int dz_2'\frac{A_2^*(s,z_2')+A_3^*(s,z_2')}{z_2'-z_2}, \tag{3.4a}$$

$$A\{s,z_1z_2+(1-z_1^2)^{\frac{1}{2}}(1-z_2^2)^{\frac{1}{2}}\cos\phi\} =\frac{1}{\pi}\int dz_3'\frac{A_2(s,z_3')+A_3(s,z_3')}{z_3'-z_1z_2-(1-z_1^2)^{\frac{1}{2}}(1-z_2^2)^{\frac{1}{2}}\cos\phi}. \tag{3.4b}$$

For simplicity we have included the absorptive parts A_2 and A_3 under the same integral sign, but they will of course contribute in different regions of the variable of integration. $A_2(s,z)$ will be nonzero only if $z<1-\{s-(M-\mu)^2\}/2q^2$, apart from a δ function at $z=1-(s-M^2-2\mu^2)/2q^2$, and $A_3(s,z)$ will be nonzero only if $z>1+2\mu^2/q^2$. The dispersion relations have been written down on the (incorrect) assumption that there are no subtractions necessary; we shall see in the following section how the theory must be modified to take them into account.

On substituting (3.4) into (3.2) and performing the integrations over z_2 and ϕ, we are left with the equation

$$A_1(s,z_1)=\frac{1}{16\pi^3}\frac{q}{W}\int dz_2'\int dz_3'\frac{1}{\sqrt{k}}\ln\frac{z_1-z_2'z_3'+\sqrt{k}}{z_1-z_2'z_3'-\sqrt{k}} \times\{A_2^*(s,z_2')+A_3^*(s,z_2')\}\{A_2(s,z_3')+A_3(s,z_3')\}, \tag{3.5}$$

[14] H. Lehmann (to be published).

where

$$k=z_1^2+z_2'^2+z_3'^2-1-2z_1z_2'z_3'. \quad (3.6)$$

We must take that branch of the logarithm which is real in the physical region $-1<z_1<1$. Equation (3.5) then gives the value of A_1 in the entire complex z_1 plane.

According to Eq. (2.16), $A(s,z_1)$ must be an analytic function of t, and therefore of z, with discontinuities of magnitude $2A_{13}$ and $2A_{12}$ as z_1 crosses the positive and negative real axes. It is easily seen that the expression for A_1 in (3.5) has this property, and, on identifying the discontinuities along the real axis with A_{13} and A_{12}, we arrive at the equations

$$A_{13}(s,z_1)=\frac{1}{8\pi^2}\frac{q}{W}\int dz_2\int dz_3\, K_1(z_1,z_2,z_3)$$
$$\times\{A_3^*(s,z_2)A_3(s,z_3)+A_2^*(s,z_2)A_2(s,z_3)\}, \quad (3.7a)$$

$$A_{12}(s,z_1)=\frac{1}{8\pi^2}\frac{q}{W}\int dz_2\int dz_3\, K_2(z_1,z_2,z_3)$$
$$\times\{A_2^*(s,z_2)A_3(s,z_3)+A_3^*(s,z_2)A_2(s,z_3)\}. \quad (3.7b)$$

The primes on z_2 and z_3 have been suppressed. K_1 and K_2 are defined by the equations

$$K_1(z_1,z_2,z_3)$$
$$=-1/[k(z_1,z_2,z_3)]^{\frac{1}{2}},\ z_1>z_2z_3+(z_2^2-1)^{\frac{1}{2}}(z_3^2-1)^{\frac{1}{2}} \quad (3.8a)$$
$$=0 \qquad z_1<z_2z_3+(z_2^2-1)^{\frac{1}{2}}(z_3^2-1)^{\frac{1}{2}}$$

$$K_2(z_1,z_2,z_3)$$
$$=1/[k(z_1,z_2,z_3)]^{\frac{1}{2}},\ z_1<z_2z_3-(z_2^2-1)^{\frac{1}{2}}(z_3^2-1)^{\frac{1}{2}} \quad (3.8b)$$
$$=0, \qquad z_1>z_2z_3-(z_2^2-1)^{\frac{1}{2}}(z_3^2-1)^{\frac{1}{2}}.$$

The points $z_1=z_2z_3\pm(z_2^2-1)^{\frac{1}{2}}(z_3^2-1)^{\frac{1}{2}}$ are the points at which k changes sign.

Let us now transform back from z to our original variables. As we shall use the dispersion relations (2.17) and (2.19), it is convenient to express A_2 and A_{12} as functions of s and s_c and A_3 and A_{13} as functions of s and t. Equations (3.7) then become

$$A_{13}(s,t)=\frac{1}{32\pi^2q^3W}\Big[\int dt_2\int dt_3$$
$$\times K_1(s;t_1,t_2,t_3)A_3^*(s,t_2)A_3(s,t_3) \quad (3.9a)$$
$$+\int ds_{c2}\int ds_{c3}\,K_1(s;t_1,s_{c2},s_{c3})A_2^*(s,s_{c2})A_2(s,s_{c3})\Big],$$

$$A_{12}(s,s_{c1})=\frac{1}{32\pi^2q^3W}\int dt_2\int ds_{c3}\,K_2(s;s_{c1},t_2,s_{c3})$$
$$\times[A_3^*(s,t_2)A_2(s,s_{c3})+A_2^*(s,s_{c3})A_3(s,t_2)]. \quad (3.9b)$$

Note that s is fixed in these equations, while s_c and t vary. K must be re-expressed as a function of the new variables by (3.3) and (2.13).

The use of Eq. (3.9), together with the dispersion relations, in order to determine the spectral functions is greatly facilitated by the fact that K is zero unless the variables satisfy certain inequalities; for all s,

$$K_1(s;t_1,t_2,t_3)=0 \quad \text{unless} \quad t_1^{\frac{1}{2}}>t_2^{\frac{1}{2}}+t_3^{\frac{1}{2}}, \quad (3.10a)$$
$$K_1(s;t_1,s_{c2},s_{c3})=0 \quad \text{unless} \quad t_1^{\frac{1}{2}}>s_{c2}^{\frac{1}{2}}+s_{c3}^{\frac{1}{2}}, \quad (3.10b)$$
$$K_2(s;s_{c1},t_2,s_{c3})=0 \quad \text{unless} \quad s_{c1}^{\frac{1}{2}}>t_2^{\frac{1}{2}}+s_{c3}^{\frac{1}{2}}. \quad (3.10c)$$

(For any particular s, the restrictions on the variables could be strengthened.) Equations (3.10) are true as long as s_{c2}, s_{c3}, t_2, and t_3 are in the regions $s_c>M^2$, $t>4\mu^2$, outside which A_2 and A_3 vanish. It follows from (3.9) that, *for any given value of* t *(or* s_c*),* $A_{13}(s,t)$ [or $A_{12}(s,s_c)$] *can be calculated in terms of* $A_3(s,t')$ *and* $A_2(s,s_c')$*, where the values of* t' *and* s_c' *involved are all less than* t *(or* s_c*).* On the other hand, by writing the dispersion relations (2.17) and (2.19) in the form

$$A_2(s,s_t)=\frac{1}{\pi}\int_{s_2(s_c)}^{\infty}ds'\,\frac{A_{12}(s',s_c)}{s'-s}$$
$$+\frac{1}{\pi}\int_{t_1(s_c)}^{\infty}dt'\,\frac{A_{23}(s_c,t')}{t'-t}, \quad (3.11a)$$

$$A_3(s,t)=\frac{1}{\pi}\int_{s_3(t)}^{\infty}ds'\,\frac{A_{13}(s',t)}{s'-s}$$
$$+\frac{1}{\pi}\int_{s_3(t)}^{\infty}ds'\,\frac{A_{23}(s',t)}{s_c'-s_c}, \quad (3.11b)$$

it is evident that $A_3(s,t)$ and $A_2(s,s_c)$ can be found in terms of $A_{12}(s',s_c)$ and $A_{13}(s',t)$, if for the moment we neglect the second term in these equations. We can therefore calculate A_{13}, A_{12}, A_3, and A_2 for all values of s and successively larger values of s_c and t. The lowest value of s_c or t for which either A_2 or A_3 is non-zero is $s_c=M^2$, at which there is a contribution of $g^2\delta(s_c-M^2)$ to A_2 from the one-nucleon state. From (3.9) and (3.10) it follows that A_{13} and A_{12} are zero if t and s_c are less than $4M^2$; for a range of values of t above this, A_{13} is nonzero and can be calculated by inserting the δ-function contribution to A_2 into (3.9a). The rest of A_2 and A_3 will still not contribute owing to (3.10). Once we have the procedure thus started, we can proceed to larger and larger values of t and s_c by alternate application of (3.9) and (3.11).[15]

Before discussing how to take the second terms of (3.11) into account, let us study in more detail the form of the functions A_{13} and A_{12} calculated thus far. In order to do this, we require the precise values of t and s_c, at a given value of s, for which the kernels K vanish; we find that

[15] It will be noticed that, though we have brought the pole in the crossing term from the one-nucleon intermediate state into our calculations, we have not yet introduced the pole in the direct term. This pole is actually a subtraction term of Eq. (2.11) and will be treated in the following section.

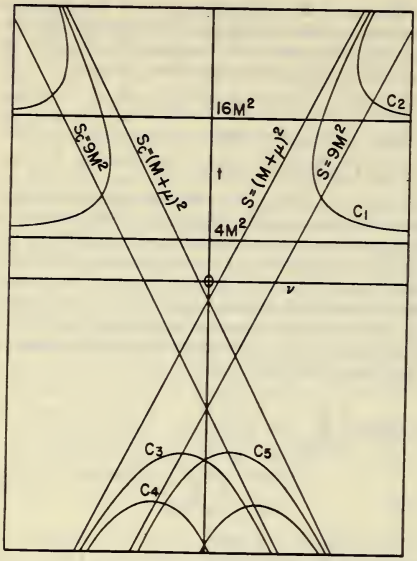

FIG. 2. Properties of the spectral functions.

$K_1(s;t_1,t_2,t_3)=0$ unless

$$t_1^{\frac{1}{2}}>t_2^{\frac{1}{2}}(1+t_3/4q^2)^{\frac{1}{2}}+t_3^{\frac{1}{2}}(1+t_2/4q^2)^{\frac{1}{2}}, \quad (3.12a)$$

$K_1(s;t_1,s_{c2},s_{c3})=0$ unless

$$t_1^{\frac{1}{2}}>(s_{c2}-u)^{\frac{1}{2}}\{1+(s_{c3}-u)/4q^2\}^{\frac{1}{2}} +(s_{c3}-u)^{\frac{1}{2}}\{1+(s_{c2}-u)/4q^2\}^{\frac{1}{2}}, \quad (3.12b)$$

$K_2(s;s_{c1},t_2,s_{c3})=0$ unless

$$(s_1-u)^{\frac{1}{2}}>t_2^{\frac{1}{2}}\{1+(s_{c3}-u)/4q^2\}^{\frac{1}{2}} +(s_{c3}-u)^{\frac{1}{2}}(1+t_2/4q^2)^{\frac{1}{2}}, \quad (3.12c)$$

where

$$u=(M^2-\mu^2)^2/s. \quad (3.13)$$

As the smallest value of s_c or t which contributes to the integrand in Eq. (3.9a) is $s_c=M^2$, where A_2 has a δ-function singularity, it follows from (3.12b) that the smallest value of t for which $A_{13}(s,t)$ is nonzero (for any given value of s) is given by

$$t^{\frac{1}{2}}=2(M^2-u)^{\frac{1}{2}}\{1+(M^2-u)/4q^2\}^{\frac{1}{2}}. \quad (3.14)$$

For very large s, this value of t approaches $4M^2$, but, as s decreases, t becomes larger and larger until, at $s=(M+\mu)^2$, it becomes infinite. Equation (3.14) has been plotted as C_1 in Fig. 2. A_{13} will be nonzero above C_1, and, near it, it will behave like $(t-t_0)^{-\frac{1}{2}}$, where t_0 is the value of t given by (3.14). It follows from (3.11b) that $A_3(s,t)$ is nonzero if $t>4M^2$, and behaves like $(t-4M^2)^{\frac{1}{2}}$ just above this limit. The value $t=4M^2$ is precisely the threshold for the process III, and we would have obtained the same results from our general reasoning in the previous section if we had neglected intermediate states containing two or more mesons but no nucleon pairs. This indicates that our assumptions are probably correct, as we have not considered the process III explicitly in this section. When we treat the subtraction terms in the dispersion relations, we shall see that A_{13} is also nonzero between $t=4\mu^2$ and $t=4M^2$, and that the region in which A_{13} is nonzero must be enlarged. The curve C_1 is therefore not yet the curve C_{13} of Fig. 1.

For a range of values of t above the curve C_1, the entire contribution to the integrand in (3.9a) comes from the δ function in A_2. At a certain point, however, the other terms in A_2 and A_3 begin to contribute. If for the moment we neglect the second term in (3.9a), the new contribution begins at the value of t obtained by putting $t_2=t_3=4M^2$ in (3.12a), since this is (at the present stage of the calculation) the lowest value of t for which A_3 is nonzero. The result has been plotted against s in Fig. 2 to give the curve C_2. As this curve approaches the line $t=16M^2$ asymptotically, there will be a corresponding new contribution to A_3 above this value, and, near it, the new contribution will behave like $(t-16M^2)^{\frac{1}{2}}$. The value $t=16M^2$ is just the threshold for the production of an additional nucleon pair in the process III, and A_3 would be expected to show such a behavior at this threshold.

We find similar discontinuities in the higher derivatives of A_{13} at series of curves (there will now be more than one for each threshold) approaching asymptotically the lines $t=4n^2M^2$, so that A_3 will have the expected behavior at the thresholds for producing n nucleon pairs.

The functions A_{12} and A_2 will exhibit the same sort of characteristics. In Eq. (3.9b), the lowest values of t_2 and s_{c3} which contribute to the integrand are $t_2=4M^2$, $s_{c3}=M^2$, so that the boundary of the region in which A_{12} is nonzero is obtained by inserting these values into (3.12c). The result is represented by the curve C_3 in Fig. 2; it approaches the line $s_c=9M^2$ as s tends to infinity. As with A_{13}, the region in which A_{12} is nonzero will be widened in the following section. From (3.19a), it follows that A_2 will (at present) be nonzero for $s_c>9M^2$, which is the threshold for pair production in the reaction II. A_{12} will also have discontinuities in the higher derivatives at series of curves such as C_4 which approach asymptotically the lines $s_c=(2n+1)^2M^2$. Finally, it can be seen that the second term of (3.9a) will give rise to further curves at which the higher derivatives of A_{13} are discontinuous, but these curves will all approach asymptotically the lines $t=4n^2M^2$.

We must now return to the second term in the Eq. (3.11), which we have so far neglected in the calculation. It can be taken into account by introducing the requirement of crossing symmetry, which has not yet been used. As in the static theory, one now has to use an iteration procedure. The function A_{23}, which only

affects the crossing term in the dispersion relation (2.5), is first neglected, and the calculation done as described. A_{23} is then found from the calculated value of A_{13} and the crossing-symmetry relations (2.23), and inserted into Eq. (3.11) for the next iteration. However, the scattering amplitude calculated by this procedure would still not satisfy the equations of crossing symmetry since, while A_{13} and A_{23} are connected by (2.23a), A_{12} does not satisfy (2.23b). We have seen that the dispersion relations together with the equation of unitarity determine A_{12} uniquely, and the result is not a symmetric function of s and s_c; even the region in which it is nonzero is not symmetric. It therefore appears that we cannot satisfy simultaneously the requirements of analyticity, unitarity (in the one-meson approximation), and crossing symmetry.

The reason why this is so is easily seen in perturbation theory. Among the graphs included in the first iteration of the one-meson approximation is Fig. 3(a). The topologically similar graph Fig. 3(b) will also be included, since Fig. 3(a) by itself would have square roots in the energy denominators and would not have the necessary analytic properties. If, therefore, crossing symmetry is to be maintained, Fig. 3(c) must also be included. In this graph, however, there is an intermediate state of a nucleon and a pair, so that the unitarity condition in the one-meson approximation is not satisfied.

This example also indicates how we should modify our iteration procedure. In addition to inserting a term A_{23}, obtained by crossing symmetry from the previous iteration, into (3.11), we must insert a term $A_{12}'(s,s_c)$ equal to $A_{12}(s_c,s)$ as calculated in the previous iteration. The contribution from this term is to be added to the contribution from $A_{12}(s,s_c)$ calculated in the normal way. A_{12}' will be nonzero above the curve C_5 in Fig. 2, and, in particular, it will be zero for all values of s_c if s is less than $9M^2$. Complete crossing symmetry is now maintained, but the addition of A_{12} violates the unitarity condition (in the one-meson approximation) for values of s greater than $9M^2$, and a perturbation expansion would include graphs such as Fig. 3(c). As these graphs will appear in higher approximations, the fact that we are forced to include them here should not be considered a disadvantage of our method. In any case, the unitarity condition is only violated where the one-meson approximation is far from correct.

The iteration procedure is found to give rise to further curves, like C_2 and C_4 (Fig. 2), at which the higher derivatives of the spectral functions are discontinuous. These new discontinuities correspond to the production of mesons together with nucleon pairs. We still do not have discontinuities at all possible thresholds.

The inclusion of the spin does not change any of the essential features of the theory, though the details are

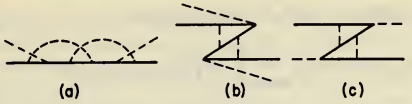

FIG. 3. Graphs which bring in intermediate states with pairs.

rather more complicated. Following Chew *et al.*,[1] we write the pion-nucleon T matrix in the form

$$T=-\frac{2\pi W}{Ew}(a+\sigma\cdot\mathbf{q}_2\sigma\cdot\mathbf{q}_1 b), \tag{3.15}$$

where E is the center-of-mass energy of the nucleon and w that of the pion. a and b are related to the quantities A and B in the expression (2.20) by the formulas

$$a=\frac{E+M}{2W}\left(\frac{A+(W-M)B}{4\pi}\right), \tag{3.16a}$$

$$b=\frac{E-M}{2W}\left(\frac{-A+(W+M)B}{4\pi}\right). \tag{3.16b}$$

The unitarity condition corresponding to (3.7) can now be worked out in terms of a and b; the equation obtained is

$$a_{13(12)}(s,z_1)=\sum_\alpha\frac{q}{\pi}\int dz_2\int dz_3\,K_{1(2)}(z_1,z_2,z_3)$$
$$\times\left\{a_\alpha^*(s,z_2)a_\alpha(s,z_3)+\frac{z_2-z_3z_1}{1-z_1^2}b_\alpha^*(s,z_2)a_\alpha(s,z_3)\right.$$
$$\left.+\frac{z_3-z_2z_1}{1-z_1^2}a_\alpha^*(s,z_2)b_\alpha(s,z_3)\right\}, \tag{3.17a}$$

$$b_{13(12)}(s,z_1)=\sum_\alpha\frac{q}{\pi}\int dz_2\int dz_3\,K_{1(2)}(z_1,z_2,z_3)$$
$$\times\left\{\frac{z_3-z_2z_1}{1-z_1^2}b_\alpha^*(s,z_2)a_\alpha(s,z_3)+\frac{z_2-z_3z_1}{1-z_1^2}\right.$$
$$\left.\times a_\alpha^*(s,z_2)b_\alpha(s,z_3)+b_\alpha^*(s,z_2)b_\alpha(s,z_3)\right\}, \tag{3.17b}$$

where $\sum_\alpha$ indicates that terms of the form $a_\alpha^*a_\alpha$ are to be replaced by $a_2^*a_2+a_3^*a_3$ in the calculation of a_{13} and b_{13} and by $a_2^*a_3+a_3^*a_2$ in the calculation of a_{12} and b_{12}, exactly as in (3.7). a_2 and b_2, a_3 and b_3, a_{12} and b_{12}, and a_{13} and b_{13} are related respectively to A_2 and B_2, A_3 and B_3, A_{12} and B_{12}, and A_{13} and B_{13} by Eqs. (3.16). The unitarity condition (3.17) can be rewritten

in terms of A and B; it then becomes

$$A_{13(12)}(s,z_1)=\sum_\alpha \frac{q}{4\pi^2 W}\int dz_2\int dz_3\, K_{1(2)}(z_1,z_2,z_3)$$
$$\times\Big\{\Big(1-\frac{w}{2W}\frac{1-z_2-z_3+z_1}{1+z_1}\Big)A_\alpha^*(s,z_2)A_\alpha(s,z_3)$$
$$+\Big(\frac{\omega}{2}\frac{1-z_2+z_3-z_1}{1-z_1}+\frac{Mw}{2W}\frac{1-z_2-z_3+z_1}{1+z_1}\Big)$$
$$\times A_\alpha^*(s,z_2)B_\alpha(s,z_3)+\Big(\frac{\omega}{2}\frac{1+z_2-z_3-z_1}{1-z_1}$$
$$+\frac{Mw}{2W}\frac{1-z_2-z_3+z_1}{1+z_1}\Big)B_\alpha^*(s,z_2)A_\alpha(s,z_3)$$
$$+\frac{W^2-M^2}{2W}\frac{1-z_2-z_3+z_1}{1+z_1}B_\alpha^*(s,z_2)B_\alpha(s,z_3)\Big\},\quad (3.18a)$$

$$B_{13(12)}(s,z_1)=\sum_\alpha \frac{q}{4\pi^2 W}\int dz_2\int dz_3\, K_{1(2)}(z_1,z_2,z_3)$$
$$\times\Big\{\frac{E}{2MW}\frac{1-z_2-z_3+z_1}{1+z_1}A_\alpha^*(s,z_2)A_\alpha(s,z_3)$$
$$+\Big(\frac{1+z_2-z_3-z_1}{2(1-z_1)}-\frac{E}{2W}\frac{1-z_2-z_3+z_1}{1+z_1}\Big)$$
$$\times A_\alpha^*(s,z_2)B_\alpha(s,z_3)+\Big(\frac{1-z_2+z_3-z_1}{1-z_1}$$
$$-\frac{E}{2W}\frac{1-z_2-z_3+z_1}{1+z_1}\Big)B_\alpha^*(s,z_2)A_\alpha(s,z_3)$$
$$+\Big(\omega-\frac{(w^2-M^2)E}{2MW}\frac{1-z_2-z_3+z_1}{1+z_1}\Big)$$
$$\times B_\alpha^*(s,z_2)B_\alpha(s,z_3)\Big\}.\quad (3.18b)$$

Equations (3.17) and (3.18) will hold separately for the amplitudes corresponding to isotopic spin $\frac{1}{2}$ and $\frac{3}{2}$.

It remains to justify the claim that the result calculated by our procedure, if expanded in a perturbation series, would give a subset of the usual perturbation series. The proof is somewhat awkward because we were unable to satisfy the unitarity condition in the one-meson approximation at all values of the energy. Let us first ignore this. The nth term in the perturbation series $A^{(n)}$ is then determined uniquely in the physical region by the following two requirements:

(i) For sufficiently small values of the momentum transfer {less than $2\mu[\frac{2}{3}(2M+\mu)/(2M-\mu)]^{\frac{1}{2}}$}, $A^{(n)}$ must satisfy the dispersion relation (2.5), a result which has been proved rigorously.[14] The absorptive part A_1 (and hence, by crossing symmetry, A_2) is known, since it is determined by unitarity in terms of lower order perturbation terms in the physical region, and by analytic continuation (with s constant) outside it.[14]

(ii) For a fixed value of s, $A^{(n)}$ is an analytic function of the momentum transfer throughout the physical region.[14]

As the functions calculated by our method certainly fulfil these requirements, they must generate the correct perturbation series.

However, our result does not satisfy the unitarity condition in the one-meson approximation at all energies, and we must examine more closely how A_1 is to be determined. Let us assume that our method gives the correct perturbation series up to the $(n-1)$th order. The reasoning developed in this section then shows that the nth-order contribution to A_1 will be of the form

$$A_1^{(n)}=\frac{1}{\pi}\int dt'\frac{A_{13}^{(n)}(s,t')}{t'-t}-\frac{1}{\pi}\int dt'\frac{A_{12}^{(n)}(s,t')}{t'-t},\quad (3.19)$$

where $A_{13}^{(n)}$ and $A_{12}^{(n)}$ are certainly zero below C_1 and above C_3, respectively, in Fig. 2. Inserting this expression into (2.5), we find that

$$A_d^{(n)}=\frac{1}{\pi^2}\int ds'\int dt'\frac{A_{13}^{(n)}(s',t')}{(s'-s)(t'-t)}$$
$$-\frac{1}{\pi^2}\int ds'\int dt'\frac{A_{12}^{(n)}(s',t')}{(s'-s)(t'-t)}.\quad (3.20)$$

The suffix d indicates that we are considering the direct and not the crossing term. The second term of (3.20) will not be an analytic function of t in the physical region, but it will have a branch point at the largest value of t for which A_{12} is nonzero. We can make it analytic by adding to A_2 the expression

$$-\frac{1}{\pi}\int dt'\frac{A_{12}^{(n)}(s_c,t')}{t'-t},\quad (3.21)$$

which we would expect from (2.17), if our representation is correct. By inserting this into (2.5) and adding the result to the second term of (3.20), we obtain

$$\frac{1}{\pi^2}\int ds'\int ds_c'\frac{A_{12}^{(n)}(s',s_c')}{(s'-s)(s_c'-s_c)},\quad (3.22)$$

which is analytic in the physical region. The contribution (3.21) to $A_2^{(n)}$ is uniquely determined from the requirement that $A^{(n)}$ be an analytic function of the momentum transfer in the physical region, and is nonzero only for $s_c>9M^2$. It corresponds to adding a graph such as Fig. 3(b) to Fig. 3(a); as A_1 for Fig. 3(c)

is nonzero for $s>9M^2$, A_2 for Fig. 3(b) will be nonzero for $s_c>9M^2$.

Finally, then, the nth-order perturbation term can be determined from the lower order perturbation terms without using any unproved properties of the scattering amplitude as follows:

(i) Calculate A_1 by unitarity, and extend it into the nonphysical region for momentum transfers less than $2\mu[\frac{2}{3}(2M+\mu)/(2M-\mu)]^{\frac{1}{2}}$ by analytic continuation.

(ii) Calculate a contribution $A_{2d}^{(n)}$ to $A_2^{(n)}$, for $s_c>9M^2$, from the requirement that if it, together with A_1, be inserted into (2.5), the resulting function $A_d^{(n)}$ must be an analytic function of the momentum transfer in the physical region. By doing this we partially include intermediate states with nucleon pairs, which is necessary if we are to maintain the required analytic properties and crossing symmetry.

(iii) Now calculate $A_2^{(n)}$ and the extra contribution to $A_1^{(n)}$ by crossing symmetry from $A_1^{(n)}$ and the extra contribution to $A_2^{(n)}$.

(iv) Find $A^{(n)}$ from (2.5) for values of the momentum transfer less than $2\mu[\frac{2}{3}(2M+\mu)/(2M-\mu)]^{\frac{1}{2}}$, and calculate it in the rest of the physical region by analytic continuation in t.

This procedure defines a one-meson approximation in perturbation theory. From what has been said, it is clear that our solution will give precisely this perturbation expansion, so that our assumptions are justified in perturbation theory.

4. SUBTRACTION TERMS IN THE DISPERSION RELATIONS

We have thus far assumed that the dispersion relations are true without any subtractions. As we have pointed out in the first section, by doing this we neglect what is physically the most important part of the scattering amplitude. In this section we shall investigate how many subtractions are necessary for each dispersion relation and shall outline how they can be calculated, leaving the details for a further paper.

Let us first consider Eqs. (2.11) and (2.16), which were used in obtaining the unitarity condition (3.9) [or (3.18) for nucleons with spin]. Even if these dispersion relations are written with subtraction terms, it is found that (3.9) is unchanged, so that the subtraction terms are only needed in the final evaluation of A from A_2 and A_3 by means of (2.11), or of A_1 from A_{12} and A_{13} by means of (2.16). The number of subtractions will depend on the behavior of A_{12}, A_{13}, A_2, and A_3, as calculated by our procedure, as s_c and t tend to infinity—we shall have to perform at least enough subtractions for (2.11) and (2.16) to converge.

It is difficult to make an estimate of the behavior of these functions at infinite values of s_c and t from the equations determining them, and we shall use indirect arguments which, though not rigorous, are very plausible. We shall find that, if the coupling constant is small enough, the functions tend to zero at infinity, so that one can write the dispersion relations without any subtractions. For larger values of the coupling constant, more and more subtractions will be needed. The reader who is prepared to accept this may omit the following two paragraphs.

We consider only the first iteration, since subsequent iterations proceed in a similar way and the results are unlikely to be qualitatively different. The result can then be expanded in a perturbation series. If the solutions obtained for this problem by other methods, such as the Tamm-Dancoff or Bethe-Salpeter methods, are expanded in a perturbation series, it is found that the series for each angular momentum state converges as long as the coupling constant is within a certain radius of convergence, and that this radius of a convergence tends to infinity with the angular momentum.[16] Our perturbation series would be different from the perturbation series obtained by these methods, partly because the intermediate states with pairs which we include are not the same as those included by either of them, and partly because, in calculating the subtraction terms (other than those at present under discussion), we shall not take into account terms corresponding to all graphs included by these approximations. Such differences would not be expected to affect qualitatively the convergence properties of the angular momentum states, and we shall assume that the results quoted above are true for our perturbation series too.

The transition amplitude for the state of total angular momentum j and orbital angular momentum $j\pm\frac{1}{2}$ can be shown to be

$$f_{j\pm}=\int_{-1}^{1}dz\,a(s,z)P_{j\pm\frac{1}{2}}(z)+\int_{1}^{1}dz\,b(s,z)P_{j\mp\frac{1}{2}}(z), \quad (4.1)$$

where a and b are the functions defined in (3.15) and (3.16). Now it is easily seen that each term in the perturbation series for $a_2(s,z)$, $a_3(s,z)$, $b_2(s,z)$, and $b_3(s,z)$ tends to zero like $1/z$ as z tends to infinity, so that the dispersion relation (2.11) for each term can be written down without any subtractions. Hence

$$f_{j\pm}^{(n)}=\int_{-1}^{1}dz\int dz'\left\{\frac{a_2^{(n)}(s,z')+a_3^{(n)}(s,z')}{z'-z}P_{j\pm\frac{1}{2}}(z)\right.$$
$$\left.+\frac{b_2^{(n)}(s,z')+b_3^{(n)}(s,z')}{z'-z}P_{j\mp\frac{1}{2}}(z)\right\} \quad (4.2)$$

$$=\int dz'\{[a_2^{(n)}(s,z')+a_3^{(n)}(s,z')]\phi_{j\pm\frac{1}{2}}(z')$$
$$+[b_2^{(n)}(s,z')+b_3^{(n)}(s,z')]\phi_{j\mp\frac{1}{2}}(z')\}, \quad (4.3)$$

[16] Note that the "potential" in the Tamm-Dancoff or Bethe-Salpeter equation involved includes only the crossing term and not the direct term, which has still to be brought into the calculation.

where

$$\phi_n(z')=\int_{-1}^{1}dz\frac{P_n(z)}{z'-z} \approx 1/z'^{n+1} \quad \text{as} \quad z'\to\infty. \tag{4.4}$$

Let us suppose that the value of the coupling constant is such that the perturbation series for states of angular momentum j_1 converges. If each term in the perturbation series for this angular momentum state is expressed by (4.3), and if we assume that we can interchange the order of summation and integration, we arrive at the equation

$$f_{j1\pm}=\int dz'\{\textstyle\sum_n[a_2^{(n)}(s,z')+a_3^{(n)}(s,z')]\phi_{j1\pm\frac{1}{2}}(z') +\sum_n[b_2^{(n)}(s,z')+b_3^{(n)}(s,z')]\phi_{j1\mp\frac{1}{2}}(z')\}. \tag{4.5}$$

In order for the integrand to exist, we see from (4.4) that a and b must be smaller than $z^{j_1-\frac{1}{2}}$ at infinite z. The dispersion relations can therefore be written down with not more than $j-\frac{1}{2}$ subtractions. In particular, if the coupling constant is small enough the dispersion relations can be written down without any subtractions.[17]

If the coupling constant is such that n subtractions are required, the unitarity condition for the states of angular momentum $\frac{1}{2}$ to $n-\frac{1}{2}$ will have to be applied separately. The wave functions for these states are polynomials of degree not greater than $n-1$ in the variable z (or s_c and t), and are not determined from the absorptive parts in the dispersion relations (2.11) and (2.16).

The calculation must be done after each iteration, as the result will be needed for the next iteration. The details of the calculation will not be discussed here, but they will in principle be similar to those of Chew and Low[4] and Dalitz, Castillejo, and Dyson,[5] and will involve considering the reciprocal of the scattering amplitude. The analytic properties of the individual angular momentum states are not as simple as in the static theory, but they can be determined from the assumed analytic properties of the transition amplitude, and, as in the static theory, the singularities not on the positive real axis can be found from the previous iteration.

The precise number of subtractions required cannot be determined without calculating the result, but it is almost certainly not less than two. It is difficult to see how the observed resonant behavior of the $P_{\frac{3}{2}}$ state could be reproduced by means of the calculations described in the last section, whereas it follows quite naturally from a Chew-Low-type calculation. If the coupling constant were large enough to bind the (3,3) resonance state, and for a certain range of values of the coupling constant below this, we would definitely have to perform two subtractions. The precise range involved is difficult to determine, but it would be expected to include those values of the coupling constant for which the (3,3) state still has the appearance of an unstable isobar. Until we state otherwise, however, we shall suppose that the coupling constant is sufficiently small for the functions $A(s,z)$ and $B(s,z)$ to tend to zero at infinite z, as the situation with regard to the other subtractions is much simpler in this case. Even then, we would have to perform one subtraction for each of A and B, since the calculations of the previous section did not include the pole of the scattering amplitude from the one-nucleon intermediate state; only the pole in the crossing term was included. The pole affects the states with $j=\frac{1}{2}$ alone, so that, if we apply the unitarity condition for these states separately by the Chew-Low method, we can include it correctly. We thereby change A and B by a quantity independent of z.

When we calculate the scattering amplitudes for the states with $j=\frac{1}{2}$, we find a ghost state in the first iteration, just as in all other models. In subsequent iterations, however, where the crossing terms contribute, it does not follow from the form of the equations that we shall necessarily find a ghost state, and, judging from the charged scalar model, we may hope that the ghost state does not in fact occur.

We now turn to consider the subtraction terms in the other dispersion relations used in the calculations, Eq. (3.11). By putting the δ-function contribution to A_2 into (3.18), it can be seen that the lowest order term in $A_{13}(s,t)$ tends to a constant as s tends to infinity, whereas the lowest order term in $B_{13}(s,t)$ behaves like $1/s$. For a certain range of values of t, only the lowest order term contributes to A_{13} and B_{13}, so that there will certainly be one subtraction in Eq. (3.11b) for A_3, while the equation for B_3 could be written down without any subtractions. We find similarly that both $A_{12}(s,s_c)$ and $B_{12}(s,s_c)$ tend to zero like $1/s$ as s tends to infinity. It would therefore appear that the dispersion relations (3.11a) did not require any subtractions. However, we have seen that $A_1(s,s_c)$ and $B_1(s,s_c)$ behave like a constant for large s_c with s constant, even for small values of the coupling constant, so that, by crossing symmetry, $A_2(s,s_c)$ and $B_2(s,s_c)$ will behave like a constant for large s. There will therefore be one subtraction term in Eqs. (3.11a) for both A_2 and B_2.

The determination of the subtraction terms in Eq. (3.11a) is not difficult, since the contributions to A_2 and B_2 from the states with $j=\frac{1}{2}$ (with the energy s_c of the reaction II kept constant) can be found by crossing symmetry from the corresponding contributions to A_1 and B_1 in the previous iteration. However, for the subtraction terms in Eq. (3.11b), we require

[17] We should emphasize that it is only in the first iteration that we relate the number of subtractions needed to the convergence of the angular momentum states. We say nothing at all about the convergence of the perturbation series in subsequent iterations, but assume simply that the behavior of the spectral functions at infinite values of z is not likely to be qualitatively different from their behavior in the first iteration.

the unitarity condition for A_3, which involves the reaction III. As there is one subtraction, only the S waves will be involved. Again we have to limit the intermediate states considered; in this first approximation we would consider the two-meson states ("two-meson approximation") and perhaps the nucleon-antinucleon intermediate states ("two-meson plus pair approximation") as well. We shall then require the meson-meson scattering amplitude (and the nucleon-antinucleon scattering amplitude if nucleon-antinucleon intermediate states are being considered). The determination of these scattering amplitudes would be as extensive a calculation as the determination of the pion-nucleon scattering amplitude, but neglect of the crossing term would probably not give rise to too great an error in our final result, in which case the S-wave amplitudes could be written down immediately in the two-meson or two-meson plus pair approximations. The meson-meson coupling constant is thereby introduced into the calculation, as has been mentioned in the introduction. Once the meson-meson and nucleon-antinucleon scattering amplitudes are known, the transition amplitude for the reaction III can be calculated. Since the integral equation is now linear, the details will be different from those of the Chew-Low calculations, but, as in their case, the solution could be written down exactly if there were no other singularities of the transition amplitude, and we can use an iteration procedure for the actual problem. The iterations will again be interspersed between the iterations of the main calculation. The S-wave portion of A_3, as calculated by this procedure, will be nonzero for $t>4\mu^2$, so that the scattering amplitude now has the expected spectral properties. The boundaries of the regions in which the spectral functions are nonzero will thereby also be changed; this will be discussed in more detail at the end of the section.

We have seen that, as long as the coupling constant is sufficiently small, we require one subtraction for each of the dispersion relations except the dispersion relation (3.11b) for B_3, for which we do not require any subtractions. It is also easily seen that this behavior is consistent—the functions as calculated in the last section, with the calculations modified by the subtraction terms, will not at any stage become too large at infinity. If, however, one were to make any additional subtractions, one would find that, on performing the calculations, one would need more and more subtractions as the work progressed, and one could not obtain any final result. The number of subtractions to be performed is therefore determined uniquely. There is one exception to this statement: we could perform one subtraction in Eq. (3.11b) for B_3. Such a subtraction is, however, excluded by the requirement that the theory remain consistent when the interaction with the electromagnetic field is introduced. If one were to make this subtraction, the scattering amplitude would behave like $f(t)\gamma(q_1+q_2)$ for large values of s. It then follows from gauge invariance that the matrix element for the processes

$$\pi^{\pm}+n\rightarrow\pi^{\pm}+n+\nu \quad \text{or} \quad \pi^{0}+p\rightarrow\pi^{0}+p+\nu$$

will contain a term which behaves like $f(t)\gamma$ for large s, where t is now minus the square of the momentum transfer of the neutral particle.[18] The contribution to B_1 and B_{13} from the $\pi-N-\gamma$ intermediate state therefore tends to infinity at least as fast as s for infinite s, so that one would require two subtractions for the dispersion relation in question and the theory would not be consistent.

Since the unitarity conditions for the two $j=\frac{1}{2}$ states of the pion-nucleon system, and for the S state of the pion-pion system, have to be applied separately by the Chew-Low method, there will be Castillejo-Dalitz-Dyson ambiguities associated with these states. The ambiguities will of course affect all states in subsequent iterations. They correspond to the existence of unstable baryons of spin $\frac{1}{2}$ and either parity, or of heavy unstable mesons of spin zero. There are no ambiguities associated with states of higher angular momentum; this is in agreement with perturbation theory, according to which it is impossible to renormalize systems containing particles of spin 1 or more. Had there been no interaction with the electromagnetic field, we could have introduced a further subtraction term which would have necessitated a separate application of the unitarity condition for the P state of the pion-pion system. The resulting Castillejo-Dalitz-Dyson ambiguity would have been associated with a heavy unstable meson of spin 1. This corresponds to the Bethe-Beard mixture of vector and scalar mesons, which can be renormalized in perturbation theory as long as there is no interaction with the electromagnetic field.

Now let us consider the situation that occurs in practice, when the coupling constant is sufficiently large for the scattering amplitude and its absorptive parts to tend to infinity with z (or s_c and t) when s remains constant. The function A_{12}' which, according to our procedure, must be added to A_{12} in iterations other than the first, will now tend to infinity with s, so that A_2, as calculated from (3.11a), would show a similar behavior. In practice, when the unitarity condition for states with $j=\frac{3}{2}$ as well as with $j=\frac{1}{2}$ must be applied separately, $A_{12}'(s,s_c)$ and $A_{23}(s_c,t)$ will tend to infinity faster than s or t, and the dispersion relation (3.11a) will require two subtractions. The subtraction terms can be determined by crossing symmetry as before. However, we have seen that, if A_2 tends to infinity with s, we cannot consistently perform the calculation, so that we shall have to introduce some further modifications.

The reason for the difficulty is probably the in-

[18] This can be shown by using a generalization of the Ward identity due to H. S. Green, Proc. Phys. Soc. (London) 66, 873 (1953), and T. D. Lee, Phys. Rev. 95, 1329 (1954), and proved by Y. Takahashi, Nuovo cimento 6, 372 (1957).

adequacy of the one-meson approximation. The breakdown occurs just at the value of the coupling constant for which the contribution to the scattering amplitude from A_{12}' is comparable to the remainder of the scattering amplitude when s is large. Since that part of A_1 calculated from A_{12}' represents a partial effect of states with one or more pairs, the contribution of these intermediate states is now important at high energies and it seems reasonable that, if one could take them into account properly, one could still perform the calculations for large values of the coupling constant. In the one-meson approximation, one would have to make some sort of a cutoff to the contribution to A_2 from the crossing term above $s=9M^2$. As this entails modifying the unitarity condition in the region where it is in any case inaccurate, it is consistent with our approximations, and it may be hoped that the theory is not very sensitive to the precise location and form of the cutoff. If one were to go to further approximations in which intermediate states with pairs were included, the cutoff would always be applied only at or above the threshold for processes which were neglected.

Once we are prepared to introduce cutoffs into our approximations, we might legitimately ask whether or not we should perform more than one subtraction in Eq. (3.11b). This could only be determined by examining the behavior of the scattering amplitude and its absorptive parts at large values of s when we go beyond the one-meson approximation. However, if A and B have the behavior assumed thus far (A remains constant and B behaves like $1/s$), the cross section would tend to zero like $1/s$ at large s, whereas the experimental results indicate that the cross section remains constant. It therefore may be necessary to perform an additional subtraction and to introduce the unitarity condition of the reaction III in P states.

At first sight it would seem as though there were Castillejo-Dalitz-Dyson ambiguities associated with all states for which the unitarity condition has to be applied separately, not only with the $j=\frac{1}{2}$ states. However, it is also possible that only the solution without any of the extra terms in the higher angular momentum waves would converge as we introduced more and more states into the unitarity equations. This solution would be an analytic continuation of the solution obtained for small values of the coupling constant, whereas the other solutions could not be continued below a certain value of the coupling constant and would have no perturbation expansion. While we can by no means exclude such a behavior, it nevertheless gives us grounds to suppose that the ambiguity exists only for meson-nucleon states with $j=\frac{1}{2}$ and for S-wave meson-meson states, even when the coupling constant is large.

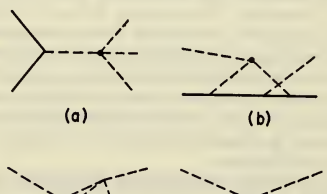

FIG. 4. Graphs involving the pion-pion interaction.

Before leaving this section, let us state the boundaries of the region in which the spectral functions A_{13}, A_{23}, and A_{12} are nonzero, i.e., the position of the curves C_{13}, C_{23}, and C_{12} in Fig. 1. Since A_3 is now nonzero for $t^2>4\mu^2$, C_{13} in the one-meson approximation is obtained by putting $t_2=t_3=4\mu^2$ in (3.12a), so that

$$t_{1a}^{\frac{1}{2}}=4\mu(1+\mu^2/q^2)^{\frac{1}{2}},$$

or

$$t_{1a}=\frac{16\mu^2(s-M^2+\mu^2)^2}{[s-(M+\mu)^2][s-(M-\mu)^2]}. \tag{4.6}$$

For any given value of s, A_{13} will be nonzero if $t>t_{1a}$. We notice that, as s tends to infinity, t_{1a} approaches the value $16\mu^2$. This is not the expected result—we have shown in Sec. 2 that it should approach the value $4\mu^2$. The reason for the discrepancy is that, in our approximation, the reaction III takes place purely through S waves for $4\mu^2<t<16\mu^2$, and A_3 will be a function only of t in this region. Had it been possible for the reaction III to go through an intermediate state of one pion, A_3 would have had a δ function at $t=\mu^2$, and, on putting this value into (3.12a), we would have obtained the expected result. As it is, however, we shall have to go beyond the one-meson approximation to get the correct boundary of A_{13}.

The reaction $N+\bar{N}\rightarrow 3\pi$ can go through a one-pion intermediate state by means of the process represented in Fig. 4(a). If, therefore, we treat the outgoing pions in the reaction $N+\pi\rightarrow N+2\pi$ as one particle with fixed energy and angular momentum, and represent the transition amplitude in the same way as we have represented the transition amplitude for pion-nucleon scattering, the absorptive part corresponding to A_3 will have a δ function at $t=\mu^2$. We can work out the resulting contribution to A_{13} (of the pion-nucleon scattering amplitude) by unitarity in the same way as we worked out the contributions from the one-meson approximation. z_2 and z_3 in Eqs. (3.4)–(3.8) will now refer to the center-of-mass deflection of the nucleon in the production reaction, and will be connected with the momentum transfer by the relation

$$z=\{q^2+q_1^2+t-[(M^2+q^2)^{\frac{1}{2}}-(M^2+q_1^2)^{\frac{1}{2}}]^2\}/2qq_1,$$

where q_1, is the center-of-mass momentum of the outgoing nucleon. The value of q_1 will depend on the relative energy of the two pions; we shall require the maximum value of q_1 (for a fixed s), which occurs when

the pions are at rest with respect to one another and is given by

$$q_{1m}^2=\{s-(M+2\mu)^2\}\{s-(M-2\mu)^2\}/4s. \quad (4.7)$$

We then find that the boundary of this contribution to A_{13} has the equation

$$t_{1b}=\frac{4\mu^2(s-M^2-2\mu)^2}{[s-(M+2\mu)^2][s-(M-2\mu)^2]}. \quad (4.8)$$

The curve represented by (4.8) approaches asymptotically the lines $t=4\mu^2$ and $s=(M+2\mu)^2$. Thus, as would be expected, this contribution to A_{13} only occurs above the threshold for pion production.

A_{13} is therefore nonzero for $t>t_1$, where

$$\begin{aligned} &t_1=t_{1a}, && (M+\mu)^2<s<(M+2\mu)^2;\\ &t_1=\min(t_{1a},t_{1b}), && (M+2\mu^2)<s<\infty; \end{aligned} \quad (4.9)$$

and $t=t_1$ is the curve C_{13} of Fig. 1. We cannot be sure that contributions from other intermediate states will not extend beyond this curve, but this is unlikely owing to the greater mass of these states.

The curve C_{23} is obtained from C_{13} simply by changing s to s_c. C_{12} can be calculated in a similar way; we find that

$$\begin{aligned} &s_{c2}=s_{c2a}, && (M+\mu)^2<s<(M+2\mu)^2\\ &\quad=\min(s_{c2a},s_{c2b}), && (M+2\mu)^2<s<\infty, \end{aligned} \quad (4.10)$$

where

$$(s_{c2a}-u)^{\frac{1}{2}}=2\mu\left\{\frac{s^2-s(3M^2+2\mu^2)+2(M^2-\mu^2)^2}{[s-(M+\mu)^2][s-(M-\mu)^2]}\right\}^{\frac{1}{2}}$$
$$+\left\{\frac{[M^2s-(M^2-\mu^2)^2][s^2-2s(M^2+3\mu^2)+(M^2-\mu^2)^2]}{s[s-(M+\mu)^2][s-(M-\mu)^2]}\right\}^{\frac{1}{2}}, \quad (4.11)$$

$$s_{c2b}(s)=s(s_{c2a}). \quad (4.12)$$

The equation $s_c=s_{c2b}$ represents in fact the boundary of the region in which A_{12}' is nonzero. We observe that, once the pion-pion interaction has been included, this region approaches asymptotically the line $s=(M+2\mu)^2$ rather than the line $s=9M^2$. The reason is that processes represented by graphs such as Fig. 4(b) are now included in our approximation, so that the crossing term will include the contribution from Fig. 4(c), the intermediate state of which involves a nucleon and two pions.

For a given real value of s, the absorptive part A of the scattering amplitude will be an analytic function of the momentum transfer as long as

$$t_2<t<t_1, \quad (4.13a)$$

where t_1 is given by (4.9), and t_2 by (4.10) and (2.13). The expansion in partial waves will converge if

$$-t_1-4q^2<t<t_1, \quad (4.13b)$$

as $-t_1-4q^2$ is always greater than t_2.

We may note finally one interesting point concerning the spectral properties of the scattering amplitude. The unitarity condition should, strictly, be used in the physical region only, and the results extended to the unphysical region by analytic continuation. This has actually been done for the reaction I, as well as for the reaction III with $t>4M^2$. For the reaction III in the region $4\mu^2<t<4M^2$, we should apply the unitarity condition with the nucleon masses taken, not on the mass shell, but at some smaller value where all the momenta would be real. The result should then be continued analytically onto the mass shell. In our case this is found to make no difference, but if, in addition to the nucleon, we had a baryon whose mass M_B satisfied the inequality

$$M_B^2<M^2-\mu^2, \quad (4.14)$$

it would be necessary to do the calculation in this way. On making the continuation to the mass shell, it would be found that the absorptive part A_3 extended below the limit $t^2=4\mu^2$. It has been shown by several workers[19] that, if an inequality such as (4.14) is satisfied, the vertex function would show similar spectral properties. The simplest graph to exhibit them in our case would be Fig. 4(d), which will obviously have properties similar to those of a vertex graph. It is thus seen that these spectral abnormalities would not limit the applicability of our method, but, on the contrary, follow from it.

5. APPROXIMATION SCHEME FOR OBTAINING THE SCATTERING AMPLITUDE

In the methods developed in the previous sections, the unitarity condition for the reaction I is satisfied for all angular-momentum states in the one-meson approximation. The unitarity condition for the reaction III is satisfied only for S states in the two-meson or two-meson plus pair approximations. The unitarity condition for higher angular momentum states of the reaction III is not satisfied, but the scattering amplitude shows the expected behavior at the threshold for competing real processes.

These properties suggest immediately a further approximation which would be consistent with our other approximations. The major portion of the work, and certainly the major part of the computing time, would be employed in calculating the spectral functions, as this involves finding double integrals which are themselves functions of two variables. The calculations would therefore be simplified if we neglected those contributions to the spectral functions which begin at the threshold for processes involving more than two particles. The only contributions to A_{13} and A_{23} left would be those beginning at $t=4M^2$, and they could be obtained by inserting the δ-function contribution to B_2

[19] Karplus, Sommerfield, and Wichman, Phys. Rev. 111, 1187 (1958); Y. Nambu, Nuovo cimento 9, 610 (1958); R. Oehme, Phys. Rev. 111, 1430 (1958).

into (3.18). The spectral function A_{12} would be zero in this approximation.

The unitarity condition for the higher angular momentum states of the reaction I is no longer satisfied. However, the terms neglected appear by their form to arise from intermediate states of the reaction III with more than two particles, so that the approximation is in the spirit of the approximations already made. We have in fact made precisely this approximation in the unitarity condition for the S waves of the reaction III. The unitarity condition for the low angular momentum states of the reaction I, and in particular for the states with $j=\frac{1}{2}$ or $\frac{3}{2}$, is still satisfied, as it has been introduced separately. The present approximation treats the reactions I, II, and III on the same footing.

To summarize, then, our method of procedure will be the following: The first few angular momentum states of A_1 and A_3 are found on the assumption that each angular momentum state is an analytic function of the square of the center-of-mass energy except for the perturbation singularities and the cuts on the positive real axis. This calculation can be done exactly if the discontinuity across the cut along the positive real axis is determined by unitarity (complications arise, as the relations connecting a and b with A and B involve square roots of kinematical factors, but the methods can be modified accordingly). A_{13} and A_{12} are also found as just described. The analytic properties of the low angular momentum states are now determined from the analytic properties of the scattering amplitude given by (2.12). The singularities can be calculated in terms of A_1, A_2, A_3. These absorptive parts can in turn be found from A_{13} and A_{23} by means of the dispersion relations (2.16), (2.17), (2.19), with subtraction terms which can be obtained from the low-angular-momentum states. In the next iteration, all the singularities of the low angular momentum states except that along the positive real axis are found from the quantities calculated in the first iteration, and the singularity along the positive real axis is redetermined from the unitarity condition. The iteration procedure is repeated until it converges. As in the calculations of Sec. 4, it is found necessary to cut off the absorptive parts A_1, A_2 and A_3 at high energies, before calculating the singularities of the low angular momentum states in the next iteration. However, the cutoff is only applied above the threshold for processes neglected in the unitarity condition, and in particular, above the threshold for pair production in the reaction I.

This approximation could be regarded as the first of a series of approximations in which more and more of the contributions to the spectral functions are included, until we ultimately reach a solution in which the unitarity condition in the one-meson approximation is satisfied for every angular momentum state. In the higher approximations the spectral functions are no longer determined by perturbation theory, but, once the contribution from the crossing term enters, they will have to be recalculated after each iteration. However, it would be more worthwhile to go beyond the one-meson approximation at the same time as we took the higher contributions to the spectral functions into account. In other words, we continue to put the reactions I, II, and III on the same footing, bringing in the higher intermediate states of all three together. If the approximation scheme converged, the exact unitarity condition of the three reactions would finally be satisfied for all angular momentum states. Needless to say, one would not in practice be able to go beyond the first one or two approximations.

The number of angular momentum states for which the unitarity condition is applied separately will, as has been explained in the last section, depend on the behavior of A and B as t (or s_c) tends to infinity with s constant. However, in our first approximation, it should be sufficient to treat separately only states with $j=\frac{1}{2}$ and $j=\frac{3}{2}$, as the other angular momentum states will not be important below the threshold for pion production. If we went beyond the one-meson approximation we would probably have to treat some higher angular momentum states separately in any case, since, for instance, two pions both in a (3,3) resonance state with a nucleon could form a $D_{\frac{5}{2}}$ state. For reaction III, one would have to treat separately S states and possibly P states as well.

If one neglected the nucleon-antinucleon intermediate state in the reaction III and only took the two-pion intermediate state into account, all three spectral functions A_{13}, A_{23}, and A_{12} would be zero, since they all begin above the threshold for processes which are being neglected. The entire scattering amplitude would then consist of "subtraction terms" for one or other of the dispersion relations. This may be the best first approximation from the point of view of the amount of work required and the accuracy of the result, as the nucleon-antinucleon intermediate state is a good deal heavier than multipion states which are being neglected. Though the spectral functions are not now brought in at all, it will of course be realized that the only justification for the approximation is that it is the first of a series of approximations which do involve the spectral functions. In this approximation, if the crossing term is neglected in the calculation of the pion-pion scattering amplitude, only intermediate S states occur in reaction III, so that the unitarity condition for the P states will not enter.

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Kinematics of General Scattering Processes and the Mandelstam Representation

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The kinematics of an arbitrary process involving two incoming and two outgoing particles is studied in terms of the invariants used in Mandelstam's representation, treating the three processes described by the same Green's function simultaneously. It is shown that the physical regions for these processes are bounded by a cubic curve in the plane of the two independent invariants. The unitarity conditions are discussed in the approximation of neglecting intermediate states of more than two particles. The formula for the spectral functions of the double dispersion relation is obtained explicitly in terms of the invariants chosen.

1. INTRODUCTION

MANDELSTAM[1] has recently proposed a representation of the scattering amplitude for meson-nucleon scattering, which is obtained from a plausible assumption about its behavior as an analytic function of two variables, the energy and momentum transfer. He has also been able to show,[2] for a more general process, that the representation is satisfied by the lower orders of the perturbation series, and that this series can actually be constructed from the representation and the unitarity relations,[3] in a two-particle approximation. In this paper we shall discuss certain aspects, mainly kinematical, of the extension of this representation to a general process. We consider together the three processes

$$\text{I}: \quad 1+2 \rightarrow 3+4,$$
$$\text{II}: \quad 1+\bar{3} \rightarrow \bar{2}+4,$$
$$\text{III}: \quad 1+\bar{4} \rightarrow \bar{2}+3.$$

The complications of spin and isotopic spin will be ignored, and all the particles will be assumed to be stable.

In Sec. 2 we shall find the physical regions for the three scattering processes in terms of the three invariants r, s, t, whose sum is equal to the sum of squared masses of the four particles. These invariants may be regarded as homogeneous coordinates in a plane, and the physical regions are then bounded by a cubic curve in this plane. The curve has three branches corresponding to the physical regions for the three scattering processes, and also a closed branch within the rst-triangle. The interior of this closed curve would correspond to the physical region for the decay process

$$\text{IV}: \quad 1 \rightarrow \bar{2}+3+4$$

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[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).
[2] S. Mandelstam, Phys. Rev. **115**, 1741 (1959).
[3] S. Mandelstam, Phys. Rev. **115**, 1752 (1959).

if this were possible. The form of the double dispersion relation for such processes has been given by Mandelstam.[2]

In Sec. 3 we shall discuss the unitarity condition for a typical process in the two-particle approximation. By making use of the determinant of scalar products of the independent momenta, the unitarity condition can be written in the form of an integral over the invariants. From this we are able to derive the relation giving the spectral functions in terms of the absorptive parts, in a similar form. The boundaries of the regions where the spectral functions are nonzero will be given by the vanishing of the determinant.

2. KINEMATICS

For convenience, we shall assume that the masses of the four particles involved in the processes I, II, III satisfy the inequalities

$$m_1 \geq m_2 \geq m_3 \geq m_4 \geq 0. \tag{1}$$

If

$$m_1 > m_2 + m_3 + m_4, \tag{2}$$

then the decay process IV is also energetically possible (although in that case we assume that it has vanishing probability). We shall choose the momenta of the incoming and outgoing particles to be p_i and $-p_i$, respectively, so that the conservation equation is always

$$p_1 + p_2 + p_3 + p_4 = 0. \tag{3}$$

The metric is chosen so that $p_i^2 = m_i^2$.

In addition to the masses, there are two independent scalar products. It is, however, convenient to use the three invariants

$$\begin{aligned} r &= (p_1+p_2)^2 = (p_3+p_4)^2, \\ s &= (p_1+p_3)^2 = (p_2+p_4)^2, \\ t &= (p_1+p_4)^2 = (p_2+p_3)^2, \end{aligned} \tag{4}$$

which satisfy

$$r + s + t = K \equiv m_1^2 + m_2^2 + m_3^2 + m_4^2. \tag{5}$$

In the center-of-mass system for process I, the momenta are $p_1 = (E_1, \mathbf{q}_1)$, $p_2 = (E_2, -\mathbf{q}_1)$, $p_3 = (-E_3, -\mathbf{q}_3)$, $p_4 = (-E_4, \mathbf{q}_3)$. The invariant r is then the square of the total energy,

$$r = W^2, \quad W = E_1 + E_2 = E_3 + E_4.$$

The magnitudes of the spatial momenta are given by

$$\begin{aligned} 4rq_1^2 &= [r - (m_1+m_2)^2][r - (m_1-m_2)^2], \\ 4rq_3^2 &= [r - (m_3+m_4)^2][r - (m_3-m_4)^2], \end{aligned}$$

and the invariants s and t may be related to the scattering angle by

$$\begin{aligned} 2s &= K - r + 4q_1q_3z - (m_1^2-m_2^2)(m_3^2-m_4^2)/r, \\ 2t &= K - r - 4q_1q_3z + (m_1^2-m_2^2)(m_3^2-m_4^2)/r, \end{aligned} \tag{6}$$

where $z = \cos(\mathbf{q}_1, \mathbf{q}_3)$.

The conditions for a physical scattering process may now be expressed in terms of r, s, t. The necessary condition

$$\pm p_i \cdot p_j > m_i m_j$$

yields

$$r > (m_1+m_2)^2 \quad \text{or} \quad r < (m_1-m_2)^2, \tag{7}$$

and similar inequalities for other pairs of masses. The requirement that the scattering angle be real can be stated in the form

$$\begin{vmatrix} p_1^2 & p_1\cdot p_2 & p_1\cdot p_3 \\ p_2\cdot p_1 & p_2^2 & p_2\cdot p_3 \\ p_3\cdot p_1 & p_3\cdot p_2 & p_3^2 \end{vmatrix} > 0.$$

This may be written as a homogeneous inequality in r, s, t,

$$rst > (r+s+t)^2(ar+bs+ct) \tag{8}$$

where the dimensionless constants a, b, c are given by

$$\begin{aligned} K^3a &= (m_1^2m_2^2 - m_3^2m_4^2)(m_1^2+m_2^2-m_3^2-m_4^2), \\ K^3b &= (m_1^2m_3^2 - m_2^2m_4^2)(m_1^2+m_3^2-m_2^2-m_4^2), \\ K^3c &= (m_1^2m_4^2 - m_2^2m_3^2)(m_1^2+m_4^2-m_2^2-m_3^2). \end{aligned}$$

The variables r, s, and t may now be regarded as homogeneous coordinates in a plane, in which the line at infinity is $r+s+t=0$. The region (8) is bounded by a cubic curve in this plane, whose asymptotes are $r=0$, $s=0$ and $t=0$. Moreover, the curve intersects its asymptotes on the line

$$ar + bs + ct = 0. \tag{9}$$

The shape of the curve is shown in Fig. 1, in which the regions marked I, II, and III are the physical regions for the corresponding processes. We note that, by the assumed inequalities for the masses, the constants a, b and c satisfy

$$a \geq b \geq c,$$

and that a and b are necessarily positive, although c may have either sign. If c is negative, the line (9) passes within the rst-triangle, and therefore the region

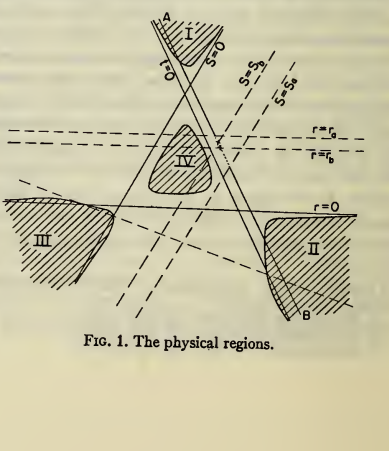

FIG. 1. The physical regions.

III in Fig. 1 includes part of this triangle. In other words, all three invariants can be positive for this process.

The bounding lines of the region defined by (7) can be shown to be tangential to the curve, so that the entire boundary of each of the physical regions is a part of the curve.[4] If the condition (2) is satisfied, then the excluded strips

$$(m_1-m_2)^2<r<(m_1+m_2)^2, \qquad (m_3-m_4)^2<r<(m_3+m_4)^2 \tag{10}$$

do not overlap, so that the region IV is allowed kinematically, as we should expect. If (2) is not satisfied, however, the strips (10) overlap, and the region IV is excluded.

3. MANDELSTAM REPRESENTATION AND UNITARITY CONDITIONS

We shall assume that for the processes in question the ordinary dispersion relations are valid. Thus for a fixed value of t, say, one can write a dispersion relation which will be an integral along a line such as AB in Fig. 1. The poles will occur on lines $r=r_b$ and $s=s_b$, and the continuous integrals will begin on lines $r=r_a$ and $s=s_a$, as indicated. Here r_b and r_a are the squared masses of the single-particle[5] and lowest two-particle intermediate states in the process I, respectively. Using the same assumptions as in his treatment of meson-nucleon scattering,[1] Mandelstam[2] has given a double-variable representation for these processes, involving three spectral functions, here denoted[6] by A_{rs}, A_{rt}, and A_{st}. The function A_{rs}, for example, will be nonzero in a region lying within the triangle defined by $r>r_a$, $s>s_a$. To find the exact boundary, C_{rs}, of this region, as shown in Fig. 2, we must use the unitarity relation. We shall assume that intermediate states with three or more particles may be neglected in this relation, and further that only one pair of particles contributes to the two-particle intermediate states. If there is more than one such pair, we have only to sum the contributions from each. In this approximation, the unitarity condition for the process I is

$$A_{1r}(rs_1)=\tfrac{1}{2}(2\pi)^{-2}\int d^4p_6 d^4p_6'\delta(p_5^2-m_5^2)\theta(p_{50})\delta(p_6^2-m_6^2) \times\theta(p_{60}')\delta(p_5+p_6-p_1-p_2)A_3^*(rs_3)A_2(rs_2), \tag{11}$$

where A_1, A_2, and A_3 now refer to the processes

$$\mathrm{I}_1:\ 1+2\to 3+4, \qquad \mathrm{I}_2:\ 1+2\to 5+6, \qquad \mathrm{I}_3:\ 3+4\to 5+6,$$

respectively. The invariants are defined[7] as in Sec. 2. We now wish to convert (11) into an integral over invariants. To do this, we perform the p_6-integration using the δ-function, and convert the p_6'-integration into one over p_5^2, $(p_5-a)^2$, $(p_5-b)^2$, $(p_5-c)^2$, where a, b and c are any three fixed timelike vectors. The Jacobian for this transformation is

$$J=\tfrac{1}{16}[-\Delta(a,b,c,p_5)]^{-\frac{1}{2}},$$

where

$$\Delta(a,b,c,p)=\begin{vmatrix} a^2 & a\cdot b & a\cdot c & a\cdot p \\ b\cdot a & b^2 & b\cdot c & b\cdot p \\ c\cdot a & c\cdot b & c^2 & c\cdot p \\ p\cdot a & p\cdot b & p\cdot c & p^2 \end{vmatrix}. \tag{12}$$

It is convenient to choose $a=p_1+p_2$, $b=p_1$, $c=-p_3$, so that

$$(p_5-a)^2=p_6^2, \quad (p_5-b)^2=s_2, \quad (p_5-c)^2=s_3.$$

Then, using the definitions of invariants, we find

$$\Delta(a,b,c,p)=\tfrac{1}{16}\begin{vmatrix} 2r & r+m_1^2-m_2^2 & r+m_3^2-m_4^2 & r+m_5^2-m_6^2 \\ r+m_1^2-m_2^2 & 2m_1^2 & m_1^2+m_3^2-s_1 & m_1^2+m_5^2-s_2 \\ r+m_3^2-m_4^2 & m_1^2+m_3^2-s_1 & 2m_3^2 & m_3^2+m_5^2-s_3 \\ r+m_5^2-m_6^2 & m_1^2+m_5^2-s_2 & m_3^2+m_5^2-s_3 & 2m_5^2 \end{vmatrix}=\Delta(r;s_1s_2s_3), \text{ say.} \tag{13}$$

This transformation is not one-to-one, since the scalar products are unaltered by changing the sign of the component of p_5 perpendicular to a, b and c. This introduces an extra factor of 2. Finally, we obtain

$$A_{1r}(rs_1)=(1/64\pi^2)\int ds_2 ds_3\,[-\Delta(r;s_1s_2s_3)]^{-\frac{1}{2}} \times A_3^*(rs_3)A_2(rs_2), \tag{14}$$

where the integration is over the region where Δ is negative.

The function Δ can of course also be expressed in terms of s_1, t_2, t_3 by interchanging m_5 and m_6 in (13), and similarly it can be expressed in terms of t_1, s_2, t_3 or t_1, t_2, s_3.

Now, in order to find an expression for the spectral functions, we have to substitute in (14) the ordinary dispersion relations for A_2 and A_3 in which r is held fixed. If we choose the value of r to be such that

$$r>(m_1+m_2)^2, \quad r>(m_3+m_4)^2, \quad r>(m_5+m_6)^2, \tag{15}$$

and take s_1 to be in the physical region for the process I_1, then it is easy to see that the condition $\Delta<0$ implies

[4] In the case of elastic scattering, the curve degenerates into a straight line and a hyperbola.

[5] Of course there may be more than one such particles and hence more than one pole, or there may be none at all.

[6] Mandelstam denotes the corresponding functions by A_{12}, A_{13}, and A_{23}. That notation would, however, be likely to cause confusion with the A_1, A_2, and A_3 introduced below.

[7] Note that if we define the signs of the momenta in I_1 and I_2 according to the convention (3), then two of the momenta in I_3 have the "wrong" sign. Thus we must define, for example, $s_3=(p_4-p_5)^2$.

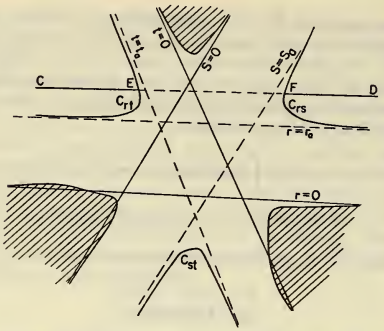

FIG. 2. The regions in which the spectral functions are nonzero.

that s_2 and s_3 are in the physical regions for the processes I_2 and I_3, respectively. Thus the denominators of the dispersion integrals for A_2 and A_3 will never vanish in the region of integration in (14), and we may ignore their small imaginary parts. For values of r which do not satisfy (15) we must have recourse to analytic continuation in the masses.[8]

It is now possible to perform the s_2 and s_3 integrations in (14) explicitly. This can be done most simply by introducing the center-of-mass variables

$$z_1=\cos(q_1,q_3),\quad z_2=\cos(q_1,q_5),\quad z_3=\cos(q_3,q_5),$$

which are linearly related to s_1, s_2, s_3 by the analogs of (6), as is done by Mandelstam.[2] We find by combining the rows and columns of the determinant (13) in a suitable way that

$$\Delta(r;s_1s_2s_3)=rq_1^2q_3^2q_5^2k(z_1z_2z_3),$$

where

$$k(z_1z_2z_3)=z_1^2+z_2^2+z_3^2-1-2z_1z_2z_3.$$

Thus the s_2 and s_3 integrations reduce to an integral already evaluated by Mandelstam.[9] The values of the spectral functions A_{1rs} and A_{1rt} may now be found from (14) by evaluating the discontinuity across the real s_1-axis. The function A_{1r} is easily seen to be an analytic function of s_1 for fixed real values of r, except for these cuts, indicated by the lines CE and FD in Fig. 2. The expression for A_{1rs}, obtained by evaluating the discontinuity along FD, may be concisely expressed in terms of the original invariants. It is[10]

$$A_{1rs}(rs_1)=(1/32\pi^2)\Big\{\int ds_2ds_3\,[\Delta(r;s_1s_2s_3)]^{-\frac{1}{2}}\times A_{3s}^*(rs_3)A_{2s}(rs_2)+\int dt_2dt_3\,[\Delta(r;s_1t_2t_3)]^{-\frac{1}{2}}\times A_{3t}^*(rt_3)A_{2t}(rt_2)\Big\}. \quad (16)$$

Here the region of integration in both terms is part of the region where $\Delta>0$, and is bounded by one branch of the curve $\Delta=0$. In the first integral, s_2 and s_3 are always positive, and in the second, t_2 and t_3 are. There are of course two other branches of the curve, corresponding to positive t_1 rather than s_1, which bound the regions where A_{1rt} is nonzero.

It should be remarked that Eq. (16) is remarkably similar to the relation (14) for A_{1r} itself, except for the fact that (14) is an integral over the physical region, whereas (16) is entirely over part of the unphysical region.

The boundary C_{rs} of the region where A_{1rs} is nonzero will clearly be given by the appropriate branch of the curve $\Delta=0$, in which the arguments s_2 and s_3 are given their minimum values, provided that these are attainable simultaneously. In the general case, we must consider all those four-cornered diagrams which are such that none of the four internal masses (of one or more particles) can be decreased.

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[8] This situation is discussed in detail in reference 2.

[9] See reference 1, Eq. (3.5).

[10] The fourth-order contribution to A_{rs} may be alternately evaluated by finding the discontinuity in A_s on crossing the real r-axis (see reference 3). The consistency of the two methods of calculation is assured by the invariance of Δ under the simultaneous interchange $m_2^2\leftrightarrow m_3^2$, $m_5^2\leftrightarrow s_2$, $m_6^2\leftrightarrow s_3$, $r\leftrightarrow s_1$. In fact Δ has a great deal more symmetry than this. It is invariant under a transitive permutation group on its ten arguments r, s_1, s_2, s_3, m_1^2, $\cdots m_6^2$, isomorphic to the symmetric group of degree 5.

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Mandelstam Representation for Potential Scattering

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A proof of the Mandelstam representation for the scattering amplitude in the case of nonrelativistic potential scattering is given for a certain class of potentials. Fredholm theory and the Lehmann representation are employed in the proof. The unitarity condition is used to provide an iteration procedure for the exact determination of the weight function appearing in the Mandelstam representation, independent of the number of subtractions required in the latter. The analytic properties of the partial wave amplitudes can be easily read off from the Mandelstam representation. It is shown that in cases where the partial wave amplitudes can be represented by dispersion integrals without subtractions, they are uniquely determined, together with the energies of any bound states, in terms of the first Born approximation and the Mandelstam weight function, which is in principle known. In these circumstances, in effect, the Mandelstam representation and unitarity, together with the first Born approximation, completely define the nonrelativistic scattering problem in a way which replaces the Schrödinger equation. Some approximation methods are discussed and compared with exact solutions for s-wave amplitudes.

I. INTRODUCTION

Field theoretic proofs of dispersion relations for scattering processes have so far been given only for restricted values of the momentum transfer. Two questions which naturally arise are as follows: is the scattering amplitude analytic in the momentum transfer variable in some larger region so that the above mentioned restriction may be lifted? Is it then possible to express the combined

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analyticity in both energy and momentum transfer in a single two-dimensional representation? The first question has been studied by Lehmann (*1*). He showed that the imaginary part of the scattering amplitude is analytic in a certain complex domain in the momentum transfer plane but this region extends no farther than the limitation already present in the derivations of the dispersion relations (*2*). No rigorous results are known about the second question. An intuitively very appealing conjecture has been made by Mandelstam (*3*), however, who assumes that the scattering amplitude is (essentially) an analytic function of both energy and momentum transfer variables with singularities only on the real axes of these two variables. This leads to a rather simple integral representation of the scattering amplitude. Mandelstam finds that the first few orders of perturbation theory indeed have this representation.

The present paper deals with the Mandelstam representation for the case of nonrelativistic potential scattering. One would hope that in this problem, where so much is known, a rigorous derivation can be carried out and various approximation methods tested against exact solutions. Another question of interest is to ask whether such a representation coupled with unitarity and perhaps certain reasonable rules can be regarded as a replacement for the Schrödinger equation.

In the case of nonrelativistic scattering by a potential with an exponential range m^{-1}, it has been shown by Khuri (*4*) (hereafter referred to as K) that a dispersion relation in the energy holds for values of the squared momentum transfer, t, less than or equal to $4m^2$. In Section II we show first by an adaptation of Lehmann's method that the scattering amplitude, aside from the first Born approximation, is in fact analytic in t regular inside an ellipse in the t-plane which intersects the real axis at values of $|t|$ not less than $4m^2$. Using the Fredholm solution of the scattering problem it is then shown that in fact the amplitude is (again excluding the Born term) analytic in the entire t plane cut from $-\infty$ to $-4m^2$. From this analyticity and that already known for the energy dependence a two-dimensional (Mandelstam) representation is established. At the end of this section the case of exchange potentials is discussed and both two dimensional and one dimensional representations established.

In Section III the two-dimensional representation is used in conjunction with unitarity to determine the weight function appearing therein by an iteration procedure. This leads to a method of approximating, to arbitrary accuracy, the subtracted scattering amplitude by a finite polynomial in the coupling constant. The sequence of polynomials thus obtained always converges provided we have supplied enough subtractions, its convergence being unaffected by the convergence or the lack of convergence of the Born series.

In Section IV the question of unitarity and bound states is considered. In the case of one bound s-state, and under the assumption that $\text{Im } f(s,t) \to g(s) \neq 0$ as $t \to \infty$, some interesting methods for calculating both the residue and the binding energy are discussed.

The analytic properties of the partial wave amplitudes are determined in Section V. At the same time we give an answer to the question posed earlier in this introduction, namely: are the Mandelstam representation and unitarity enough to define the nonrelativistic scattering problem? The answer to the above question is as follows. We show in this section that all the partial waves with $l \geqq n + 1$, where $(n + 1)$ is the number of subtractions in the Mandelstam representation, are determined by an integral over the in principle known weight function. For $l < n + 1$ we write the partial wave amplitudes as a ratio of two functions and then obtain a nonsingular Fredholm type equation for these functions. The kernel in these equations is given in terms of the Mandelstam weight function; but we cannot rule out that undetermined subtraction constants will appear. The weight function is determined in principle by the method of Section III. When no subtractions are required for the partial waves all the phase shifts are uniquely determined no matter what n is.

II. ANALYTICITY IN MOMENTUM TRANSFER AND THE MANDELSTAM REPRESENTATION

A. Restrictions on Potentials

Our discussion will be limited to static central potentials. Certain assumptions have already been made about the potential, V, in K and these we retain; namely,

$$\begin{aligned} &| V(r) | < M/r^2, \\ &\int_{M'}^{\infty} r^2\, dr\, | V(r) | < \infty, \\ &\int_{0}^{M''} dr\, r\, | V(r) | < \infty, \end{aligned} \tag{2.1}$$

where M, M', M'' are finite positive numbers. These restrictions suffice to guarantee that V has a three-dimensional Fourier transform and that the Fredholm series solution developed in K shall apply. There is one further assumption which plays a critical role in our work, namely, that our potentials have the representation

$$rV(r) = \int_0^{\infty} d\mu\, \sigma(\mu)\, e^{-\mu r}. \tag{2.2}$$

If $V(r)$ has a range m^{-1}; namely, if $\int_0^{\infty} dr\, r\, | V(r) |\, e^{\alpha r} < \infty$, for $0 \leqq \alpha \leqq m$; then $\sigma(\mu)$ is zero for $\mu < m$. The conditions on rV which insure the possibility of the representation (2.2) are (5) (assuming $\sigma(\mu)$ bounded almost everywhere except perhaps for δ-function singularities): (i) $rV(r)$ has derivatives of all

orders $(0 < r < \infty)$ and (ii) $|\, d^k(rV)/dr^k \,| < (\text{const.})\ k!/r^{k+1}$ for

$$k = 0, 1, 2, \cdots \quad (0 < r < \infty).$$

The second and third inequalities in (2.1) imply, respectively, that $\sigma(\mu)/\mu \to 0$ as $\mu \to 0$ and that $\sigma(\mu) \to 0$ as $\mu \to \infty$.

In discussing the analytic properties of the scattering amplitude it is convenient to isolate the first Born approximation and discuss it separately. This is given by

$$f_B(t) = \frac{-M}{2\pi\hbar^2} \int d\mathbf{r}\, e^{-i\boldsymbol{\Delta}\cdot\mathbf{r}} V(r) = \frac{-2M}{\hbar^2} \int_0^\infty d\mu\, \frac{\sigma(\mu)}{\mu^2 + t}, \tag{2.3}$$

where $\boldsymbol{\Delta} = \mathbf{k}_f - \mathbf{k}_i$, $t = \Delta^2$, and $\mathbf{k}_f$, $\mathbf{k}_i$ are the final and initial wave number vectors, respectively; M is the reduced mass of the system. Regarded as a function of t, $f_B(t)$ is evidently analytic in the t plane cut along the negative real axis from $-\infty$ to $-m^2$, where we recall that the effective lower limit on the second integral in (2.3) is $m \geqq 0$. In exceptional cases, where $\sigma(\mu)$ is a linear combination of δ-functions or derivatives thereof, $f_B(t)$ may have isolated poles on the negative real axis rather than a branch cut. Since $\sigma(\mu) \to 0$ as $\mu \to \infty$, $f_B(t) \to 0$ as $t \to \infty$.

B. The Lehmann Method in Nonrelativistic Theory

It is convenient to choose units such that $\hbar^2/2M = 1$, so that the energy becomes simply k^2 which we designate by s. The scattering amplitude is then a function of the two scalar variables $s = k^2$ and $t = \Delta^2$ and is called $f(s,t)$. Using Lehmann's method we shall show that $f'(s,t) \equiv f(s,t) - f_B(t)$ is, for fixed real $s > 0$, analytic in an ellipse in the t plane which includes the physical region. As in the derivation of the usual dispersion relation in K causality plays no evident role, in contrast to the field theoretic situation. Here the result emerges almost kinematically.

The scattering amplitude satisfies the familiar Lippmann–Schwinger integral equation

$$f(\mathbf{k}_f, \mathbf{k}_i) = f_B(\mathbf{k}_f - \mathbf{k}_i) - \int \frac{d\mathbf{k}'}{(2\pi)^3} \frac{\tilde{V}(\mathbf{k}_f - \mathbf{k}')f(\mathbf{k}', \mathbf{k}_i)}{k'^2 - k^2 - i\epsilon}, \tag{2.4}$$

where $\tilde{V}$ is the Fourier transform of the potential; in the present notation and units the first Born approximation (2.3) is given by $f_B(q) = -\tilde{V}(q)/4\pi$. We may write an explicit formal solution to this equation, namely

$$f(\mathbf{k}_f, \mathbf{k}_i) = f_B(\mathbf{k}_f, \mathbf{k}_i) + \int d\mathbf{q}_1 \int d\mathbf{q}_2\, \tilde{V}(\mathbf{k}_f - \mathbf{q}_1) G(\mathbf{q}_1, \mathbf{q}_2; k^2) \tilde{V}(\mathbf{q}_2 - \mathbf{k}_i), \tag{2.5}$$

where $G(\mathbf{q}_1, \mathbf{q}_2; k^2)$ is essentially the Fourier transform of the full Green's function:

$$G(\mathbf{q}_1, \mathbf{q}_2; k^2) = \frac{-1}{4\pi} \int \frac{d\mathbf{r}_1}{(2\pi)^3} \int \frac{d\mathbf{r}_2}{(2\pi)^3} e^{-i\mathbf{q}_1 \cdot \mathbf{r}_1} \cdot \left\langle \mathbf{r}_1 \left| \frac{1}{k^2 + i\epsilon + \nabla^2 - V} \right| \mathbf{r}_2 \right\rangle e^{i\mathbf{q}_2 \cdot \mathbf{r}_2}. \tag{2.6}$$

Using our representation for the potential we see that the difference $f - f_B \equiv f'$ may be written as

$$f'(\mathbf{k}_f, \mathbf{k}_i) = \int d\mu_1 \int d\mu_2 \, \sigma(\mu_1)\sigma(\mu_2) \int d\mathbf{q}_1 \int d\mathbf{q}_2 \cdot \frac{1}{\mu_1^2 + (\mathbf{k}_f - \mathbf{q}_1)^2} G(\mathbf{q}_1, \mathbf{q}_2; k^2) \frac{1}{\mu_2^2 + (\mathbf{q}_2 - \mathbf{k}_2)^2}. \tag{2.7}$$

Following Lehmann, we choose a special coordinate system to effect some of the integrations in (2.7):

$$\begin{aligned} k_i &= k(1, 0, 0), \\ k_f &= k(\cos\theta, \sin\theta, 0), \\ q_j &= q_j (\sin\beta_j \cos\alpha_j, \sin\beta_j \sin\alpha_j, \cos\beta_j); \quad j = 1, 2. \end{aligned} \tag{2.8}$$

Evidently θ is the scattering angle given by $\cos^{-1}(\mathbf{k}_i \cdot \mathbf{k}_f / k^2)$. We now introduce the new variables

$$\begin{aligned} \chi &= \alpha_1 - \alpha_2, \\ \lambda_j &= \frac{\mu_j^2 + k^2 + q_j^2}{2kq_j \sin\beta_j}, \qquad j = 1, 2. \end{aligned} \tag{2.9}$$

It is easy to show that for fixed k the minimum value, λ_0, of λ_j, taken where μ_j^2, q_j, and β_j vary over their respective domains of integration in (2.7), is given by

$$\lambda_0 = (1 + m^2/k^2)^{1/2}, \tag{2.10}$$

where again m^{-1} is the "range" of our potential. With these new variables we may obtain from (2.7) the following representation for $f'(\mathbf{k}_f, \mathbf{k}_i)$,

$$f'(k, \cos\theta) = \int_{\lambda_0}^{\infty} d\lambda_1 \int_{\lambda_0}^{\infty} d\lambda_2 \int_0^{2\pi} d\chi \cdot \int_0^{2\pi} d\alpha_1 \frac{w(\lambda_1, \lambda_2, k, \chi)}{[\lambda_1 - \cos(\theta - \alpha_1)][\lambda_2 - \cos(\alpha_1 - \chi)]}, \tag{2.11}$$

where w is a weight function which need not be specified further here. The integration over α_1 may be carried out easily and we find

$$f'(k, \cos\theta) = 2\pi \int_{\lambda_0}^{\infty} d\lambda_1 \int_{\lambda_0}^{\infty} d\lambda_2 \int_0^{2\pi} d\chi \, w(\lambda_1, \lambda_2, k, \chi) \cdot \frac{\lambda_1/\sqrt{\lambda_1^2 - 1} + \lambda_2/\sqrt{\lambda_2^2 - 1}}{\lambda_1\lambda_2 + (\lambda_1^2 - 1)^{1/2}(\lambda_2^2 - 1)^{1/2} - \cos(\theta - \chi)}. \tag{2.12}$$

Finally we introduce $y = \lambda_1\lambda_2 + [(\lambda_1^2 - 1)(\lambda_2^2 - 1)]^{1/2}$ as a new variable and obtain

$$f'(k, \cos\theta) = \int_{y_0}^{\infty} dy \int_0^{2\pi} d\chi \, \frac{\bar{w}(y, k, \chi)}{y - \cos(\theta - \chi)}, \tag{2.13}$$

with $\bar{w}$ a new weight function and y_0 the minimum value for y for fixed k. It is evident that

$$y_0 = 2\lambda_0^2 - 1 = 1 + 2m^2/k^2.$$

Since an identical representation holds for f'^*, we may express both Re f' and Im f' in a similar manner. Lehmann's argument may be taken over bodily at this point and it leads to the following conclusion. Both Re f' and Im f' are, for given real $k^2 \geqq 0$, analytic functions of $\cos\theta$ regular inside an ellipse in the $\cos\theta$ plane centered at the origin with semi-major axis $y_0 = 1 + 2m^2/k^2$ and semi-minor axis $(y_0^2 - 1)^{1/2}$. In terms of $t = 2k^2(1 - \cos\theta)$ one has analyticity inside an ellipse which intersects the real axis at $t = -4m^2$ and $t = 4m^2 + 4k^2$. The absolute minimum, $4m^2$, is just the value below which the proof of the ordinary dispersion relation given in K holds.

It is interesting to note that our results hold for both the real and imaginary parts of f' whereas Lehmann found a larger region for Im f' than for Re f'. It is possible that use of the analog of (2.5) in field theory, which would be the Low equation written out as a sum over states, would yield a similar result.

C. Analyticity in the Cut t-Plane[1]

We now show that the domain of analyticity in the t-plane is larger than that obtained in the previous section. To do this we have to look in more detail at the structure of the full Green's function. The procedure consists of studying the

[1] Preprints of the first three main sections of this paper were completed and first circulated in the summer of 1959. After that we learned that the analyticity in the $\cos\theta$-plane has also been proved by T. Regge independently. His technique involves the use of complex angular momenta, and his proof establishes the absence of essential singularities at infinity in the $\cos\theta$-plane, a fact that we have assumed in our paper. J. Bowcock and A. Martin have also independently shown the analyticity in $\cos\theta$ but only for each term in the Born series.

Fredholm series solution of the scattering integral equation (2.4). Each term is shown to be analytic in t regular in the cut t-plane with the cut extending from $-\infty$ to $-4m^2$ along the negative real axis. In Appendix I we prove that the series converges uniformly in any closed finite region in the t-plane that does not include a point on the cut.

The Fredholm solution of (2.4) may be written as

$$f(\mathbf{k}_f, \mathbf{k}_i) = f_B(\mathbf{k}_f, \mathbf{k}_i) + \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{N(\mathbf{k}_f, \mathbf{p}; k)}{D(k)} f_B(\mathbf{p} - \mathbf{k}_i), \tag{2.14}$$

where N and D are given by uniformly convergent series expansion (*6*). For potentials satisfying (2.1) the series for $D(k)$ converges uniformly and for real values of $k^2 > 0$ never vanishes. Since $D(k)$ does not depend on the directions of $\mathbf{k}_f$ and $\mathbf{k}_i$ it plays no role in our proof of the analyticity in t.

The numerator N has the well-known expansion

$$N(\mathbf{k}_f, \mathbf{p}; k) = \frac{1}{(2\pi)^3} \frac{\tilde{V}(\mathbf{k}_f, \mathbf{p})}{p^2 - k^2 - i\epsilon} + \sum_{n=1}^{\infty} (-1)^n \frac{(2\pi)^{-3n}}{n!} \frac{1}{p^2 - k^2 - i\epsilon}$$

$$\times \int \prod_{i=1}^{n} \frac{d\mathbf{p}_i}{p_i^2 - k^2 - i\epsilon} \begin{vmatrix} \tilde{V}(\mathbf{k}_f, \mathbf{p}) & \tilde{V}(\mathbf{k}_f, \mathbf{p}_1) & \cdots & \tilde{V}(\mathbf{k}_f, \mathbf{p}_n) \\ \tilde{V}(\mathbf{p}_1, \mathbf{p}) & 0 & \tilde{V}(\mathbf{p}_1, \mathbf{p}_2) & \cdots \\ \vdots & & 0 & \\ \tilde{V}(\mathbf{p}_n, \mathbf{p}) & \tilde{V}(\mathbf{p}_n, \mathbf{p}_1) & \cdots & 0 \end{vmatrix}, \tag{2.15}$$

where $\tilde{V}(\mathbf{p}_i, \mathbf{p}_j) = \tilde{V}(\mathbf{p}_i - \mathbf{p}_j)$. This series converges uniformly for physical $\mathbf{k}_f$, $\mathbf{p}$, and k as can be seen by converting it term by term into the x-space series of Jost and Pais (*6*). Consider now the nth term in the above expansion. The general term in the development of the determinant will, after relabeling of the indices, have the general form

$$N_l^{(n)}(\mathbf{k}_f, \mathbf{p}; k) = \int \prod_{j=1}^{l} \frac{d\mathbf{p}_j}{p_j^2 - k^2 - i\epsilon} \cdot \frac{F_l^{(n)}(k)}{p^2 - k^2 - i\epsilon} \tilde{V}(\mathbf{k}_f, \mathbf{p}_1)\tilde{V}(\mathbf{p}_1, \mathbf{p}_2) \cdots \tilde{V}(\mathbf{p}_{l-1}, \mathbf{p}_l)\, \tilde{V}(\mathbf{p}_l, \mathbf{p}), \tag{2.16}$$

where $l \leqq n$ and $F_l^{(n)}(k)$ is obtained from the integrations over the remaining variables, $\mathbf{p}_{l+1}, \mathbf{p}_{l+2}, \cdots, \mathbf{p}_n$.

Inserting for the $\tilde{V}$'s our representation we find that (2.16) leads to a contribution to $f' = f - f_B$ which may be written as follows:

$$f_l'^{(n)}(\mathbf{k}_f, \mathbf{k}_i) = \frac{1}{(2\pi)^3} \frac{F_l^{(n)}(k)}{D(k)} \int \prod_{i=1}^{l+1} \frac{\sigma(\mu_i)\, d\mu_i\, dp_i}{2[p_i^2 - k^2 - i\epsilon]} \frac{\sigma(\mu_{l+2})\, d\mu_{l+2}}{2k^2}$$
$$\times \int d\Omega_1 \cdots d\Omega_{l+1} [\lambda_1 - \hat{k}_f \cdot \hat{p}_1]^{-1} [\lambda_2 - \hat{p}_1 \cdot \hat{p}_2]^{-1} \cdots [\lambda_{l+2} - \hat{p}_{l+1} \cdot \hat{k}_i]^{-1}, \tag{2.17}$$

where $\hat{\mathbf{k}}_f$, $\hat{\mathbf{k}}_i$, and $\hat{\mathbf{p}}_j$ are unit vectors and the λ_j are defined by

$$\lambda_j = \frac{\mu_j^2 + p_{j-1}^2 + p_j^2}{2p_{j-1}p_j}, \qquad 1 < j < l + 2,$$
$$\lambda_1 = \frac{\lambda_1^2 + k^2 + p_1^2}{2kp_1}, \tag{2.18}$$
$$\lambda_{l+2} = \frac{\mu_{l+2}^2 + p_{l+1}^2 + k^2}{2p_{l+1}k}.$$

The next step is to carry out the angular integrations. Here one encounters the basic integral

$$I = \int d\Omega_{\hat{\mathbf{p}}} [\tau_1 - \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}]^{-1} [\tau_2 - \hat{\mathbf{p}} \cdot \hat{\mathbf{p}}_2]^{-1}, \tag{2.19}$$

with τ_1, $\tau_2 \geqq 1$. This may be evaluated (for example by Feynman methods) and the result written as

$$I = (4\pi) \int_{\eta_0}^{\infty} d\eta \frac{1}{\eta - \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2} \cdot \frac{1}{\mathrm{K}(\eta)}, \tag{2.20}$$

where

$$\eta_0 = \tau_1\tau_2 + (\tau_1^2 - 1)^{1/2} (\tau_2^2 - 1)^{1/2},$$
$$\mathrm{K}(x) = [(\tau_1\tau_2 - x)^2 - (\tau_1^2 - 1)(\tau_2^2 - 1)]^{1/2}. \tag{2.21}$$

By repeated integrations we see that $f_l'^{(n)}(k, \cos\theta)$ can be brought to the form

$$f_l'^{(n)}(k, \cos\theta) = \int_{\eta_{0l}^{(n)}}^{\infty} d\eta \frac{\phi_l^{(n)}(\eta, k)}{\eta - \cos\theta}, \tag{2.22}$$

and it is readily seen that

$$\eta_{0l}^{(n)} \geqq 1 + \frac{2m^2}{k^2}. \tag{2.23}$$

This same result could have also been obtained by using the Lehmann techniques which we employed in getting from (2.7) to (2.12). A similar analysis and representation holds for the complex conjugate of $f'^{(n)}_l$. Hence the representa-

tion (2.22) holds for the real and imaginary parts separately of each term in the Fredholm expansion.[2] From (2.22) one can easily conclude that, for fixed k, each term in the Fredholm expansion for $f'(k, \cos\theta)$ is analytic in $\cos\theta$ regular in the whole $\cos\theta$-plane except for a cut on the positive real axis extending from $\cos\theta = 1 + 2m^2/k^2$ to infinity.

The remainder of the argument hinges on the uniformity of convergence of the Fredholm series for any finite region in the $\cos\theta$-plane not including the cut. This is taken up in Appendix I, and we conclude here that the whole $f' = f - f_B$ is analytic in the cut $\cos\theta$-plane. Finally, regarding f' as a function of $k^2 = s$ and $t = 2s(1 - \cos\theta)$ we conclude that it is analytic in the t-plane cut from $t = -\infty$ to $t = -4m^2$.

D. The Mandelstam Representation

It is now a relatively simple matter to extend the ordinary dispersion relation, proved in K only for real $t \leqq 4m^2$, to all finite complex t anywhere in the cut plane excluding the cut. The Mandelstam representation will then follow from an application of Cauchy's theorem in the t-plane.

The familiar dispersion relation for the scattering amplitude $f(s,t)$ [recall $s = k^2$, $t = 2s(1 - \cos\theta)$] is

$$f(s,t) = f_B(t) + \sum_{i=0}^{r} \frac{\Gamma_i(t)}{s + s_i} + \frac{1}{\pi}\int_0^\infty ds' \frac{\operatorname{Im} f(s', t)}{s' - s}, \qquad \operatorname{Im} s \neq 0. \quad (2.24)$$

The s_i are the (negative) energies of the bound states in our units and the $\Gamma_i(t)$ are polynomials in t, the degree being l_i, the angular momentum of the ith bound state.

We have shown, however, that $\operatorname{Im} f(s', t)$ for real $s' \geqq 0$ is analytic in the t-plane cut along the negative real axis from $-\infty$ to $-4m^2$ [note that $f_B(t)$ is real]. The residues Γ_i are simple polynomials in t. Hence we see that the last two terms on the right in (2.24) can be extended into the cut t-plane to define an $f'(s,t)$ which will be an analytic function of two complex variables, s and t, regular in the region defined by the topological product of the two cut planes

[2] Actually the situation is slightly more complicated because of the singular nature of the Green's functions $[p^2 - k^2 - i\epsilon]^{-1}$. For example in the case of the second Born approximation for a Yukawa potential of range m^{-1} one has

$$f_2 \sim \int_{1+(2m^2/s)}^{1+2m^2/s+m^4/2s^2} \frac{d\eta}{\eta - \cos\theta} \frac{1}{[m^4 - 2s\{s(\eta - 1) - 2m^2\}]^{1/2}}$$

$$+ i\int_{1+2m^2/s+m^4/2s^2}^{\infty} \frac{d\eta}{\eta - \cos\theta} \frac{1}{[2s\{s(\eta - 1) - 2m^2\} - m^4]^{1/2}},$$

with $s = k^2$. Thus the cut for the real part extends over the real axis from $1 + (2m^2/s)$ to $1 + 2m^2/s + m^4/2s^2$, and that for the imaginary part picks up where the real part leaves off.

(except, of course, for the poles in the s-plane at the bound states, s_i). That the $f'(s,t)$ thus obtained by continuation of the right side of (2.24) is identical with the actual $f'(s,t)$ can be easily ascertained when $m \neq 0$. From the definition of the original $f'(s,t)$ one can show that there is a domain of the variables s,t for which $f'(s,t)$ is regular in both. This domain includes the line $0 \leqq t \leqq 4m^2$ on the real axis of the t plane and any line just above the positive s-axis starting at Re $s > m^2$. In that case the two functions are the same in the finite interval $0 \leqq t \leqq 4m^2$ on the real axis, with any $s = s' + i\delta$ and $s' > m^2$, hence by analyticity everywhere. The case $m = 0$ requires special treatment. The main steps are sketched in a footnote.[3] The same conclusion obtains, namely, Eq. (2.24) is true for all t in the cut plane.

The remaining step to a definite integral representation for $f(s,t)$ consists simply in writing for Im $f(s', t)$ a dispersion representation which exhibits the proven analyticity in t. The only uncertainty in so doing stems from our lack of knowledge of the asymptotic behavior of Im $f(s', t)$ for large t. The simplest assumption, namely that Im $f(s', t) \to 0$ as $|t| \to \infty$, is certainly inconsistent with unitarity, as we shall show, if there are bound states and maybe even if there are not. (All this is in spite of the fact, which one can easily deduce from (2.22), that each term in the Fredholm expansion for $f'(s,t)$ vanishes as $|t| \to \infty$.) In order to cover all contingencies we write

$$\text{Im } f(s', t) = (-1)^n \frac{t^{n+1}}{\pi} \int_0^\infty dt' \frac{\rho(s', t')}{t'^{n+1}(t' + t)} + \sum_{a=0}^{n} \frac{t^j}{j!} g_k(s'), \qquad (2.25)$$

where n, the degree of the polynomial, is not specified at present and $g_j(s')$ is the jth derivative of Im $f(s', t)$ evaluated at $t = 0$. The limit of integration in the above integral has been written formally as zero; it in general depends on s' but is never less than $4m^2$. The actual domain in the s,t plane over which ρ is nonvanishing will be determined by unitarity.

A general form for the Mandelstam representation which we study for the rest of this paper is

$$f(s,t) = f_B(t) + \sum_{i=1}^{r} \frac{\Gamma_i(t)}{s + s_i} + (-1)^n t^{n+1} \int_0^\infty \frac{ds'}{\pi} \int_0^\infty \frac{dt'}{\pi} \frac{\rho(s', t')}{t'^{n+1}(t' + t)(s' - s - i\epsilon)} + \sum_{j=0}^{n} t^j \int_0^\infty ds' \frac{g_j(s')}{s' - s - i\epsilon}. \qquad (2.26)$$

[3] One starts by defining the function f_ϵ as

$$f_\epsilon(\mathbf{k}_f, \mathbf{k}_i) = f_B(t) - \frac{1}{4\pi} \int \frac{d\mathbf{x}}{(2\pi)^3} \frac{d\mathbf{y}}{(2\pi)^3} e^{-i\mathbf{k}_f \cdot \mathbf{x}} V(x) e^{-\epsilon x} G(\mathbf{x},\mathbf{y}; k) V(y) e^{-\epsilon y} e^{i\mathbf{k}_i \cdot \mathbf{y}}.$$

for $0 \leqq t \leqq 4\epsilon^2$. We now use the same reasoning as above to extend (2.24) for all values of t in the cut plane. One then has to study the limit of the right-hand side of (2.24) as $\epsilon \to 0$. The limit of the residues $\Gamma_{i\epsilon}(t)$ has to be dealt with separately.

It is worth noting that there is no need to make, for fixed t, subtractions in s, because it has been shown in K that $f(s,t) \to f_B(t)$ as $s \to \infty$. We also remark that $n \geqq \bar{l}_i$ where $\bar{l}_i$ is the largest angular momentum appearing in the bound states. Otherwise one can easily show that (2.26) will be inconsistent with unitarity.

E. Exchange Potential

Before going on to consider unitarity and the iteration scheme we wish to show how the representation is modified if there are exchange potentials. We shall also derive the generalization of the one dimensional dispersion relation for this case. We assume now

$$\begin{aligned} V(x) &= V_e(x) + V_2(x)\, P_x \\ &\equiv V_e(x) + V_0(x), \end{aligned} \tag{2.27}$$

where P_x is the space exchange operator. Note that $\langle \mathbf{x} \mid V_0 \mid \mathbf{y} \rangle = V_2(x)\delta(\mathbf{x} + \mathbf{y})$. In order to simplify the discussion we shall assume that there are no bound states and insert their contribution explicitly at the end.

As usual the total Green's function satisfies the following integral equation

$$G(\mathbf{x},\mathbf{y};k) = g(\mathbf{x},\mathbf{y};k) + \int g(\mathbf{x},\mathbf{z};k)[V_e(z) + V_0(z)]G(\mathbf{z},\mathbf{y};k)\, d\mathbf{z}, \tag{2.28}$$

where

$$g(\mathbf{x},\mathbf{y};k) = \frac{-1}{4\pi} \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}. \tag{2.29}$$

In operator notation we can write (2.28) as

$$G = g + g(V_e + V_0)\, G. \tag{2.30}$$

Let us define R as,

$$R = (V_e + V_0)G; \tag{2.31}$$

then R will satisfy the following integration equation

$$R = (K_e + K_0) + (K_e + K_0)R, \tag{2.32}$$

with

$$K_e = V_e g; \qquad K_0 = V_0 g. \tag{2.33}$$

We now write

$$R = R_e + R_0\,, \tag{2.34}$$

where R_e and R_0 are defined by the following integral equations:

$$\begin{aligned} R_e &= K_e + (K_e R_e + K_0 R_0) \\ R_0 &= K_0 + (K_e R_0 + K_0 R_e). \end{aligned} \tag{2.35}$$

This system of equations is obviously consistent with (2.32). One can resort to matrix notation to write (2.35) as one equation,

$$R_i = K_i^{(0)} + \sum_{j=1}^{2} K_{ij} R_j , \tag{2.36}$$

where $R_1 \equiv R_e$, $R_2 \equiv R_0$, $K_1^{(0)} = K_e$, $K_2^{(0)} = K_0$, $K_{11} = K_{22} = K_e$, and $K_{12} = K_{21} = K_0$. Each of the elements K_{ij} and $K_i^{(0)}$ are analytic in k regular in the upper half plane. In this case a sufficient condition for the existence of solutions R_i of (2.36) which are regular in the upper half k plane is

$$\left| \sum_{j=1}^{2} \int K_{ij}(\mathbf{x},\mathbf{z}) K_{jl}(\mathbf{z},\mathbf{y})\, d\mathbf{z} \right| \leqq \text{const.} \left| \frac{V_i(x)}{x} \right| ; \qquad \text{Im } k \geqq 0. \tag{2.37}$$

For if (2.37) holds, it follows from Ref. K that we can represent R_i by Fredholm series which are uniformly convergent in the upper half k plane. The conditions (2.1) for both V_e and V_2 are sufficient to obtain (2.37) by the methods of Ref. K.

The scattering amplitude is given by

$$\begin{aligned} f(\mathbf{k}_f, \mathbf{k}_i) = \frac{-1}{4\pi} \tilde{V}_e(\mathbf{k}_f, \mathbf{k}_i) + \frac{-1}{4\pi} \tilde{V}_0(\mathbf{k}_f, \mathbf{k}_i) \\ + \int e^{-i\mathbf{k}_f \cdot \mathbf{x}} R(\mathbf{x},\mathbf{y}; k)[V_e(y) + V_0(y)] e^{i\mathbf{k}_i \cdot \mathbf{y}}\, d\mathbf{x}\, d\mathbf{y}. \end{aligned} \tag{2.38}$$

We write

$$f = f_e + f_0 , \tag{2.39}$$

where

$$f_e = \frac{-1}{4\pi} \langle \mathbf{k}_f \mid V_e \mid \mathbf{k}_i \rangle + \langle \mathbf{k}_f \mid R_e V_e + R_0 V_0 \mid \mathbf{k}_i \rangle, \tag{2.40}$$

$$f_0 = \frac{-1}{4\pi} \langle \mathbf{k}_f \mid V_0 \mid \mathbf{k}_i \rangle + \langle \mathbf{k}_f \mid R_0 V_e + R_e V_0 \mid \mathbf{k}_i \rangle. \tag{2.41}$$

It is worth pointing out that the split of the amplitude into an even and an odd part as done by Hamilton (7), who was the first to discuss this point, is not correct in its details.

Using (2.35) and the lemma proved in the appendix of Ref. K, one can show that for any ϵ there is a k_0 such that for any k with $|k| \geqq k_0$ and Im $k = \kappa \geqq 0$, the following two inequalities hold,

$$|R_e(\mathbf{x},\mathbf{y};k) - K_e(\mathbf{x},\mathbf{y};k)| < \epsilon\, e^{-\kappa|\mathbf{x}-\mathbf{y}|}\,\frac{|V(x)|}{x}, \tag{2.42}$$

$$|R_0(\mathbf{x},\mathbf{y};k) - K_0(\mathbf{x},\mathbf{y};k)| < \epsilon\, e^{-\kappa|\mathbf{x}+\mathbf{y}|}\,\frac{|V(x)|}{x}. \tag{2.43}$$

These last two inequalities coupled with (2.40) and (2.41) are enough to show that $f_e(k^2, t)$ is analytic in k regular in the upper half plane when $t = (\mathbf{k}_f - \mathbf{k}_i)^2$ is fixed and $t \leqq 4m^2$. Similarly $f_0(k^2, u)$ is analytic in k in the upper half plane for fixed $u = (\mathbf{k}_f + \mathbf{k}_i)^2$ and $u \leqq 4m^2$. Hence we can now write two dispersion relations

$$\begin{aligned} f_e(s,t) &= f_e^{(B)}(t) + \frac{1}{\pi}\int_0^\infty ds'\,\frac{\operatorname{Im} f_e(s';t)}{s' - s - i\epsilon}, \\ f_0(s,u) &= f_0^{(B)}(u) + \frac{1}{\pi}\int_0^\infty ds'\,\frac{\operatorname{Im} f_0(s';u)}{s' - s - i\epsilon}. \end{aligned} \tag{2.44}$$

As usual no subtractions are needed in this case since as $|s| \to \infty$ $f_e(s,t) \to f_e^{(B)}(t)$ and $f_0(s,u) \to f_0^{(B)}(u)$.

We now turn to the question of the analytic properties of $f_e(s,t)$ and $f_0(s,u)$ as functions of u and t for fixed real s.

If we substitute Eqs. (2.35) in (2.40) and (2.41) we obtain the following integral equations for f_e and f_0 in momentum space

$$\begin{aligned} f_e(\mathbf{k}_f,\mathbf{k}_i) &= f_e^{(B)}(\mathbf{k}_f,\mathbf{k}_i) + \int [\bar{K}_e(\mathbf{k}_f,\mathbf{p}) f_e(\mathbf{p},\mathbf{k}_i) + \bar{K}_0(\mathbf{k}_f,\mathbf{p}) f_0(\mathbf{p},\mathbf{k}_i)]\, d\mathbf{p}, \\ f_0(\mathbf{k}_f,\mathbf{k}_i) &= f_0^{(B)}(\mathbf{k}_f,\mathbf{k}_i) + \int [\bar{K}_e(\mathbf{k}_f,\mathbf{p}) f_0(\mathbf{p},\mathbf{k}_i) + \bar{K}_0(\mathbf{k}_f,\mathbf{p}) f_e(\mathbf{p},\mathbf{k}_i)]\, d\mathbf{p}, \end{aligned} \tag{2.45}$$

where

$$\bar{K}_{e,0}(\mathbf{k}_f,\mathbf{p}) = \frac{1}{(2\pi)^3}\,\frac{\tilde{V}_{e,0}(\mathbf{k}_f,\mathbf{p})}{p^2 - k^2 - i\epsilon}. \tag{2.46}$$

As in (2.36) we can rewrite Eqs. (2.45) in matrix notation

$$f_i = f_i^{(B)} + \sum_{j=1}^{2} \bar{K}_{ij} f_j, \tag{2.47}$$

with $f_1 = f_e$, $f_2 = f_0$, $\bar{K}_{11} = \bar{K}_{22} = \bar{K}_e$, $\bar{K}_{12} = \bar{K}_{21} = \bar{K}_0$. We can now follow

the arguments of part C and write the Fredholm solution for each f_1. Noting that

$$\tilde{V}_0(\mathbf{p}_1, \mathbf{p}_2) = \int \frac{\sigma_0(\mu)\, d\mu}{\mu^2 + (\mathbf{p}_1 + \mathbf{p}_2)^2},$$
$$\tilde{V}_e(\mathbf{p}_1, \mathbf{p}_2) = \int \frac{\sigma_e(\mu)\, d\mu}{\mu^2 + (\mathbf{p}_1 - \mathbf{p}_2)^2}, \tag{2.48}$$

one can easily check that each term in the series for $f'_1(k, \cos\theta)$ has a representation corresponding to (2.22)

$$f'^{(n)}_1(k, \cos\theta) = \int_{\eta_0}^{\infty} d\eta \, \frac{\phi_1^{(n)}(\eta,k)}{\eta - \cos\theta}, \qquad \eta_0 = 1 + \frac{2m^2}{k^2}. \tag{2.49}$$

On the other hand each term in the series for f'_2 will have the representation

$$f'^{(n)}_2(k, \cos\theta) = \int_{\eta_0}^{\infty} d\eta \, \frac{\phi_2^{(n)}(\eta,k)}{\eta + \cos\theta}. \tag{2.50}$$

The difference between (2.49) and (2.50) arises from the fact that the integrand on the right-hand side of (2.17) will have only an even number of factors $[\lambda_j + \hat{\mathbf{p}}_{j-1}\cdot\hat{\mathbf{p}}_j]^{-1}$ when we consider the series for f'_1, whereas an odd number of factors $[\lambda_j + \hat{\mathbf{p}}_{j-1}\cdot\hat{\mathbf{p}}_j]^{-1}$ will appear when we consider each term for f'_2.

The argument given in the Appendix will still hold for this case. We then conclude that $f'_e(k, \cos\theta)$ is analytic in $\cos\theta$ regular in the cut plane with the cut extending from $\cos\theta = 1 + (2m^2/k^2)$ to $+\infty$, whereas $f'_0(k, \cos\theta)$ has the cut on the negative real axis from $\cos\theta = -(1 + 2m^2/k^2)$ to $-\infty$. The same result holds for both the real and imaginary parts of f'_e and f'_0. Assuming that there are no essential singularities as $|\cos\theta| \to \infty$ we can apply the Cauchy theorem in both the t and u plane and obtain from (2.44)

$$f_e(s,t) = f_e^{(B)}(t) + \frac{1}{\pi^2}\int_0^{\infty} ds' \int_{4m^2}^{\infty} dt' \, \frac{\rho_e(s', t')}{(s' - s - i\epsilon)(t' + t)}$$
$$f_0(s,u) = f_0^{(B)}(u) + \frac{1}{\pi^2}\int_0^{\infty} ds' \int_{4m^2}^{\infty} du' \, \frac{\rho_0(s', u')}{(s' - s - i\epsilon)(u' + u)}, \tag{2.51}$$

where

$$t = 2s(1 - \cos\theta); \qquad u = 2s(1 + \cos\theta);$$
$$u + t - 4s = 0. \tag{2.52}$$

In (2.51) we have not written down subtractions in both t and u which might appear.

The bound-state terms appear in the final amplitude from the discrete poles of the total Green's function for negative energy. The same poles appear for both $f_e(s,t)$ and $f_0(s,u)$. The easiest way to see this is by noting that these poles will arise from the zeros of the Fredholm denominator in the solution of (2.47). That denominator is the same for both f_e and f_0. The final representation has the form (aside from subtractions terms which must appear but which we don't write explicitly for the moment)

$$\begin{aligned} f(s,t,u) &= f_e^{(B)}(t) + f_0^{(B)}(u) + \sum_i \frac{\Gamma_i(t)}{s + s_i} \\ &+ \int_0^\infty \frac{ds'}{\pi} \int_{4m^2}^\infty \frac{dt'}{\pi} \frac{\rho_e(s',t')}{(s' - s - i\epsilon)(t' + t)} \\ &+ \int_0^\infty \frac{ds'}{\pi} \int_{4m^2}^\infty \frac{du'}{\pi} \frac{\rho_0(s',u')}{(s' - s - i\epsilon)(u' + u)}, \end{aligned} \tag{2.53}$$

where the residues $\Gamma_i(t)$ are defined by

$$\begin{aligned} \Gamma_i(t) = \frac{1}{4\pi} \Big\{ \Big[\int d\mathbf{x}\, e^{-i\mathbf{k}_f \cdot \mathbf{x}} (V_e + V_0)\psi_i(\mathbf{x}) \Big] \\ \cdot [d\mathbf{x}\, \psi_i^*(\mathbf{x})(V_e + V_0) e^{i\mathbf{k}_i \cdot \mathbf{x}}] \Big\}_{s=-s_i} \end{aligned} \tag{2.54}$$

We are regarding $u(= 4s - t)$ as an auxiliary variable; hence the residues Γ_i are functions of t alone.

Our final task is to find the one-dimensional dispersion relation which is satisfied in this case. To do this one expresses the second denominator in terms of s and t, using $t + u = 4s$. Then make a partial fraction decomposition:

$$\frac{1}{(s' - s)} \cdot \frac{1}{(u' + u)} = \left[\frac{1}{s' - s} + \frac{4}{u' + 4s - t} \right] \frac{1}{u' - t + 4s'}. \tag{2.55}$$

The first term is then of the standard form, whereas the second one is something new. After several trivial variable changes we obtain

$$\begin{aligned} f(s,t) = f_e^{(B)}(t) + f_0^{(B)}(4s - t) + \sum_i \frac{\Gamma_i(t)}{s + s_i} + \int_0^\infty \frac{ds'}{\pi} \frac{\rho_+(s',t)}{s' - s} \\ + \int_{m^2}^\infty \frac{ds'}{\pi} \frac{\rho_-(s',t)}{s' + s - t/4}, \end{aligned} \tag{2.56}$$

where the weight functions ρ_+, ρ_- are given by

$$\begin{aligned} \rho_+(s',t) &= \int_{4m^2}^\infty \frac{dt'}{\pi} \frac{\rho_e(s',t')}{(t' + t)} + \int_{m^2}^\infty \frac{d\tau'}{\pi} \frac{\rho_0(s',4\tau')}{\tau' + s' - t/4}, \\ \rho_-(s',t) &= \int_0^\infty \frac{d\tau'}{\pi} \frac{\rho_0(\tau',4s')}{s' + \tau' - t/4}. \end{aligned} \tag{2.57}$$

The fear that one encounters trouble when $t > 4m^2$ in that the cut which for smaller t lies on the negative s axis crosses over to $+s$ is groundless. The apparent complexity of ρ_+ is just cancelled in this case by a contribution from ρ_-. If we limit ourselves to ordinary Yukawa potentials for both V_e and V_2 then (2.56) is almost identical with the nonrelativistic form of the nucleon-nucleon dispersion relations derived from field theory by Goldberger *et al.* (*8*). Note that in this paper our units are such that $\hbar^2/2M = 1$ with M the reduced mass of the system.

III. THE UNITARITY CONDITION

The simplest case to be considered is that of no bound states. We shall now show how unitarity can be used to determine the weight function ρ of (2.26). The procedure that we shall go through does not change with the number of subtractions in t. Hence for simplifying the algebra we shall assume that

$$\operatorname{Im} f(s,t) \to 0 \quad \text{as} \quad |t| \to \infty.$$

This assumption is at least consistent with unitarity when there are no bound states.

In this case the Mandelstam representation becomes

$$f(s,t) = f_B(t) + \int_0^\infty \frac{ds'}{\pi} \int_0^\infty \frac{dt'}{\pi} \frac{\rho(s',t')}{(s' - s - i\epsilon)(t' + t)}, \tag{3.1}$$

where we recall that

$$f_B(t) = -\int_0^\infty d\mu \frac{\sigma(\mu)}{\mu^2 + t}. \tag{3.2}$$

Since the analyticity properties have been specified completely and the potential has been given, we would expect the unitarity condition to enable us to find the weight function ρ and thus the solution to the problem. We write this condition as

$$\operatorname{Im} f(s,t) = \frac{\sqrt{s}}{4\pi} \int d\Omega' \, f^*[s, (\mathbf{k}_f - \mathbf{k}')^2] f[s, (\mathbf{k}' - \mathbf{k}_i)^2], \tag{3.3}$$

where $t = (\mathbf{k}_f - \mathbf{k}_i)^2, \mathbf{k}_f^2 = \mathbf{k}_i^2 = \mathbf{k}'^2 = s$. Obviously this relation is valid only in the physical region $t \leqq 4s$ and must be given meaning for larger t by analytic continuation. When (3.1) is substituted into (3.3) we encounter in every term the integral

$$I = 4s^2 \int d\Omega' \frac{1}{[t_1 + (\mathbf{k}_f - \mathbf{k}')^2][t_2 + (\mathbf{k}' - \mathbf{k}_i)^2]}, \tag{3.4}$$

with the notation $\tau_1 = 1 + (t_1/2s)$, $\tau_2 = 1 + t_2/2s$, and $\hat{\mathbf{k}}' = \mathbf{k}'/k'$, etc., we can write

$$I = \int d\Omega' \frac{1}{[\tau_1 - \hat{\mathbf{k}}_f \cdot \hat{\mathbf{k}}'][\tau_2 - \hat{\mathbf{k}}' \cdot \hat{\mathbf{k}}_i]}. \tag{3.5}$$

This integral is then the same one encountered in Section II and for our purposes is most conveniently written as

$$I = 4\pi \int_{t_0}^{\infty} \frac{dt'}{t' + t} \frac{1}{\mathrm{K}(1 + t'/2s)}, \tag{3.6}$$

where $t_0 = 2s[\tau_1\tau_2 - 1 + \{(\tau_1^2 - 1)(\tau_2^2 - 1)\}^{1/2}]$ and K is defined in (2.21). From (3.1) we have

$$\mathrm{Im}\, f(s,t) = \frac{1}{\pi} \int_0^{\infty} dt' \frac{\rho(s,t')}{t' + t}. \tag{3.7}$$

Using (3.6), we find from (3.3) an equation for $\rho(s,t)$:

$$\begin{aligned} \rho(s,t) = {} & \int d\mu_1\, \sigma(\mu_1) \int d\mu_2\, \sigma(\mu_2) K(s,t;\mu_1^2,\mu_2^2) \\ & - 2P \int_0^{\infty} \frac{ds_1}{\pi} \int_0^{\infty} \frac{dt_1}{\pi} \frac{\rho(s_1,t_1)}{s_1 - s} \int d\mu_2\, \sigma(\mu_2) K(s,t\,;t_1,\mu_2^2) \\ & + \int_0^{\infty} \frac{ds_1}{\pi} \int_0^{\infty} \frac{dt_1}{\pi} \int_0^{\infty} \frac{ds_2}{\pi} \int_0^{\infty} \frac{dt_2}{\pi} \frac{\rho(s_1,t_1)\rho(s_2,t_2)}{[s_1 - s + i\epsilon][s_2 - s - i\epsilon]} \\ & \cdot K(s,t;t_1,t_2), \end{aligned} \tag{3.8}$$

where

$$K(s,t;t_1,t_2) = \frac{\pi}{2} \frac{\theta\left[t - t_1 - t_2 - \dfrac{t_1t_2}{2s} - \dfrac{(t_1t_2)^{1/2}}{2s}\{16s^2 + 4s(t_1 + t_2) + t_1t_2\}^{1/2}\right]}{[s\{t - (t_1^{1/2} + t_2^{1/2})^2\}\{t - (t_1^{1/2} - t_2^{1/2})^2\} - tt_1t_2]^{1/2}}. \tag{3.9}$$

The function $K(s,t;t_1,t_2)$, regarded as a function of s,t, fails to vanish in a region bounded by the curve t_1t_2/s, for small s, and the line $t = (t_1^{1/2} + t_2^{1/2})^2$ for large s.

It is evident from (3.8) and (3.9) that if there is no lower limit on the μ_i, i.e., $m = 0$, $\rho(s,t)$ will be different from zero in the entire first quadrant of the s,t plane. This follows from the fact that the first term in (3.8) [which is the second Born approximation for ρ] already shows this. We shall not consider the $m = 0$ problem any further.

Consider the case $m \neq 0$. It is easy to see that the true limiting curve in the s,t plane (i.e., the curve below which $\rho(s,t)$ vanishes) is determined by the second Born approximation. This term is nonvanishing when $K(s,t;m^2,m^2) \neq 0$. This leads to the condition

$$t > 4m^2 + m^4/s. \tag{3.10}$$

The second and third terms on the right-hand side of (3.8) contribute to ρ only

in a region entirely inside that determined by (3.10), since in those terms t_1 and t_2 are always greater than m^2. With these observations in mind we can see how in this case of $m \neq 0$ one can construct ρ in a sequence of well-defined steps. Thus (3.10) determines an absolute minimum value of t [below which $\rho(s,t)$ vanishes] given by $4m^2$. The second term in (3.8) will therefore contribute to ρ only in the region where $K(s,t;\, 4m^2,\, m^2) \neq 0$. This describes a boundary curve asymptotic to $t = 4m^4/s$ (small s) and $t = 9m^2$ (large s). The third term in (3.8) contributes only above a curve asymptotic to $t = 16m^4/s$ (small s) and $t = 16m^2$ (large s). There is evidently then a finite region in the s,t plane where $\rho(s,t)$ is given *exactly* by the second Born approximation. [Mathematically the area lies between the curves $s(t - 4m^2) - m^4 = 0$, and $s(t - 9m^2)\,(t - m^2) = 4m^2t$.]

It is clear then that knowing ρ over this region enables us to construct ρ in a still larger region by substituting into the second term of (3.8) and integrating over the region where ρ is known exactly. There are also contributions from the third term which ultimately contribute. Before this happens, however, a new region of finite size will be generated over which ρ is known exactly. At this stage ρ consists of terms proportional to λ^2 and λ^3, where we have introduced a coupling strength parameter, λ, to characterize the number of times the potential enters in each term. At the next step we will have terms proportional to λ^2, λ^3, and λ^4. What we are doing is gradually filling out the whole allowable s,t plane and expressing $\rho(s,t)$ by a sequence of polynomials in λ which increases in order as we move out. In any finite region of the allowed s,t plane ρ is given exactly by a polynomial in λ of finite degree. The degree of the polynomial increases as the region moves out toward infinity. Having obtained ρ in this manner we can imagine substituting it into (3.1) and carrying out the integration over the s,t plane at each step integrating over the regions where ρ is known exactly.

This leads to an expression for $f(s,t)$ which is the limit of a sequence of polynomials. If our assumption about the behavior of $\operatorname{Im} f(s,t)$ as $|\, t \,| \to \infty$ is correct, then the sequence converges. It is not unusual to be able to represent a function by a convergent sequence of polynomials even if a power series representation diverges.

If $\operatorname{Im} f(s,t)$ does not vanish as $|\, t \,| \to \infty$ then we have to use a subtracted form of (3.1)

$$F(s,t) \equiv f(s,t) - f_B(t) - \sum_{j=0}^{n} \frac{t^j}{j!}\, g_j(s) = \frac{t^{n+1}}{\pi^2} \int ds' \int dt' \, \frac{\rho(s',\, t')}{(s' - s - i\epsilon)t'^{n+1}(t' + t)}\,. \tag{3.11}$$

An analogous procedure could be followed to find ρ in this case and (3.8) would again obtain. This allows us to approximate $F(s,t)$ arbitrarily closely by a sequence of polynomials in λ. This sequence will always converge to $F(s,t)$ pro-

viding we have supplied enough subtractions in (3.11). We stress here that this convergence does not depend at all on whether the usual Born series for $f(s,t)$ converges or not.

To determine the amplitude $f(s,t)$, we still have to find the functions $g_j(s)$. Instead of doing this directly we shall in Section V consider the partial waves separately. There we show that for $l > n$ the partial waves are given by integrals over ρ and hence are determined by the iteration outlined above. For $l \leqq n$ we obtain for the partial wave amplitudes nonsingular integral equations, which have unique solutions. The kernels in these equations are determined by ρ.

We note here that the procedure outlined in this section for determining ρ could be easily generalized to determine ρ_e and ρ_0 in the exchange potential case. At every step in the iteration one can easily separate the contributions to ρ_e and to ρ_0. It is worth mentioning that the iterative construction of ρ is quite analogous to, though considerably simpler than, that encountered by Mandelstam in the relativistic case.

IV. UNITARITY AND BOUND STATES

Before treating the general case in Section V we shall consider in this section the case with a bound state where $\operatorname{Im} f(s', t) \to g(s') \neq 0$ as $|t| \to \infty$. This situation has several amusing consequences.

In this case we can only have a bound s-state for otherwise our assumption about $\operatorname{Im} f(s', t)$ would be inconsistent with unitarity. For definiteness we shall assume that we have a bound s-state with energy $-s_0$. The Mandelstam representation in this case will be

$$\begin{aligned} f(s,t) &= f_B(t) + \int_0^\infty \frac{ds'}{\pi} \int_0^\infty \frac{dt'}{\pi} \frac{\rho(s', t')}{(s' - s - i\epsilon)(t' + t)} + \frac{\Gamma}{s + s_0} \\ &\quad + \int_0^\infty \frac{ds'}{\pi} \frac{g(s')}{s' - s - i\epsilon} \\ &\equiv f^{(0)}(s,t) + L_+(s), \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} f^{(0)} &= f_B(t) + \int_0^\infty \frac{ds'}{\pi} \int_0^\infty \frac{dt'}{\pi} \frac{\rho(s', t')}{(s' - s - i\epsilon)(t' + t)}, \\ L_+(s) &= L(s + i\epsilon) = \Gamma/(s + s_0) + \int_0^\infty \frac{ds'}{\pi} \frac{g(s')}{s' - s - i\epsilon}. \end{aligned} \tag{4.2}$$

We note that $f^{(0)} \to 0$ as $|t| \to \infty$ so that L_+ is the limiting value of f at infinite t. All the partial wave amplitudes obtained from (4.1) by projection will, except for the s wave, be the same as those obtained from an unsubtracted representation. In the next section we will see that they are determined by quadra-

tures when ρ is known. We wish to emphasize that if $\Gamma \neq 0$, the term involving $g(s)$ *must* be present in order to avoid a blatant violation of unitarity. It may be true that g must be present even if there are no bound states. In any case it is easy to show that the iterative construction of ρ given previously is not affected by the presence of L. From this it follows that $f^{(0)}$ satisfies unitarity by itself:

$$\operatorname{Im} f^{(0)}(s,t) = \frac{\sqrt{s}}{4\pi} \int d\Omega' \, f^{(0)*}[s, (\mathbf{k}_f - \mathbf{k}')^2] f^{(0)}[s, (\mathbf{k}' - \mathbf{k}_i)^2]. \quad (4.3)$$

If we now substitute (4.1) into the unitarity condition for the full amplitude and use (4.3), we find

$$\operatorname{Im} L_+(s) = 2\sqrt{s} \operatorname{Re} [f_0^{(0)} L_+^*] + \sqrt{s} \, | L_+(s) |^2, \qquad s > 0, \quad (4.4)$$

where $f_0^{(0)}$ is the s-wave projection of $f^{(0)}$, namely

$$f_0^{(0)} = \frac{1}{4\pi} \int d\Omega \, f^{(0)} = \frac{e^{i\delta^0} \sin \delta^0}{\sqrt{s}}. \quad (4.5)$$

The fact that $f_0^{(0)}$ can be represented this way with a *real* δ^0 follows from a comparison between (4.4) and the unitarity condition for the full s-wave amplitude which is

$$\operatorname{Im} f_0^{(0)} + \operatorname{Im} L_+ = 2\sqrt{s} \operatorname{Re} [f_0^{(0)} L_+^*] + \sqrt{s} \, | L_+ |^2 + \sqrt{s} \, | f_0^{(0)} |^2; \quad (4.6)$$

thus $\operatorname{Im} f_0^{(0)} = \sqrt{s} \, | f_0^{(0)} |^2$. If we introduce $L_-(s) \equiv L(s - i\epsilon) = L_+^*(s)$, we find that it satisfies

$$\operatorname{Im} L_-(s) = -2\sqrt{s} \operatorname{Re} [f_0^{(0)} L_-] - \sqrt{s} \, | L_-(s) |^2, \qquad s > 0. \quad (4.7)$$

We are confronted here with a simple mapping problem, namely to find a function $L(s)$ which goes to zero at infinity, is analytic in the s plane except for a cut along the positive real axis with values on the cut given by (4.4) and (4.7), and which has a pole at $s = -s_0$.

To solve this problem it is convenient to introduce $L(s) = 1/K(s)$ in terms of which (4.4) and (4.7) become

$$\begin{aligned} \operatorname{Im} K_+(s) &= -\sqrt{s} - 2\sqrt{s} \operatorname{Re} [f_0^{(0)} K_+(s)], \\ \operatorname{Im} K_-(s) &= \sqrt{s} + 2\sqrt{s} \operatorname{Re} (f_0^{(0)} K_-^*(s)]. \end{aligned} \quad (4.8)$$

Using the expression for $f_0^{(0)}$ in terms of δ^0, we may cast (4.8) into the form

$$K_\pm(s) = \mp \frac{i\sqrt{s}}{\cos 2\delta^0} + [1 \mp i \tan 2\delta^0] \operatorname{Re} K_\pm(s), \quad (4.9)$$

which holds along the positive real s axis. This inhomogeneous mapping problem may be solved in terms of the solution of the corresponding homogeneous one.

We define this lattter problem as

$$K_{\pm}(s) = [1 \mp i \tan 2\delta^0] \operatorname{Re} K_{\pm}(s), \qquad s > 0, \text{ real.} \tag{4.10}$$

This has the well-known solution

$$\begin{aligned} K_{\pm} &= P(s) \exp\left[-\frac{1}{\pi}\int_0^{\infty} ds' \frac{2\delta^0(s')}{s' - s_+ - i\epsilon}\right] \\ K_{\pm} &= P(s) e^{-p(s) \mp 2i\delta^0(s)}, \end{aligned} \tag{4.11}$$

where $P(s)$ is an arbitrary polynomial which is assumed to have no zeros for s real and positive and we have assumed $\delta^0(0) \leqq \pi/2$. Further we have defined

$$p(s) = P\int_0^{\infty} \frac{ds'}{\pi} \frac{2\delta^0(s')}{s' - s}. \tag{4.11a}$$

Using (4.10) and (4.9) we may write

$$\frac{K_+}{K_+} - \frac{K_-}{K_-} = -2i\sqrt{s}e^{p(s)}/P(s), \qquad s > 0, \text{ real.} \tag{4.12}$$

The solution to this problem is standard. We find

$$K_+(s) = -K_+(s)\int_0^{\infty} \frac{ds'}{\pi} \frac{\sqrt{s'}\, e^{p(s')}}{P(s')(s' - s - i\epsilon)}. \tag{4.13}$$

To this solution, of course, may be added any entire function.

Before discussing this result further let us see that it automatically yields a unitary total s-wave amplitude. We define for positive s a real function G by

$$e^{2i\delta^0}K_+(s) = -G(s) - i\sqrt{s}, \tag{4.14}$$

and recalling that $e^{i\delta}\sin\delta/\sqrt{s} = f_0^{(0)} + 1/K_+(s)$, where δ is the full s-wave phase shift, using (4.14) we find the manifestly unitary relation

$$e^{2i\delta} = e^{2i\delta^0}\frac{G(s) - i\sqrt{s}}{G(s) + i\sqrt{s}}. \tag{4.15}$$

In treating the solution to our mapping problem, (4.13), there are two courses open to us, one conservative and somewhat unsatisfactory, one daring and rather conjectural. The conservative procedure is to choose the polynomial $P(s)$ so as to insure the simple pole at $s = -s_0$ and thus abandon all hope of computing the energy s_0. We write

$$K_+(s) = -\frac{(s + s_0)}{\pi} e^{-p(s) - 2i\delta^0}\int_0^{\infty} ds \frac{\sqrt{s'}\, e^{p(s')}}{(s' + s_0)(s' - s - i\epsilon)}. \tag{4.16}$$

From this we see that the residue Γ and, for example, the exact scattering length

a, are given by

$$\frac{1}{\Gamma} = -\frac{1}{\pi}\int_0^\infty ds' \frac{\sqrt{s'}\, e^{[p(s')-p(-s_0)]}}{(s'+s_0)^2},$$
$$a = a_0 - \left[\frac{s_0}{\pi}\int_0^\infty \frac{ds'\, e^{[p(s')-p(0)]}}{\sqrt{s'}\,(s'+s_0)}\right]^{-1}, \tag{4.17}$$

where a_0 is the scattering length associated with δ^0. Thus, given s_0 and δ^0 (the latter obtainable in principle from the iteration scheme) we have a complete solution to our problem. We have tested these relations by guessing at δ^0 and examining the consequences. We try first $\tan\delta^0 = \sqrt{s}\, a_0$. It is easy to show that

$$p(s) + 2i\,\delta^0(s) - p(0) = -2\ln(1 + a_0\sqrt{-s}) \tag{4.18}$$

and the integrals involved in computing Γ and a are trivial. They lead to

$$\Gamma = -2\sqrt{s_0},$$
$$a = a_0 - \left[\frac{\sqrt{s_0}}{1+\sqrt{s_0}a_0}\right]^{-1} = -1/\sqrt{s_0}\,. \tag{4.19}$$

The result for Γ agrees exactly with a direct evaluation from the Schrödinger equation with a scattering length potential; that for a is the correct zero range limit for the scattering length and amusingly enough has the correct sign independent of our guess, a_0. The $\sqrt{s_0}$ dependence of Γ is a general characteristic of our solution (4.17) independent (essentially) of δ^0 and further this dependence on $\sqrt{s_0}$ is rigorously true for any potential for small s_0.

A slightly more complicated model which has the phase shift behave quite differently for large s is $\tan\delta^0(s) = a_0\sqrt{s}/(1 + \frac{1}{2}ra_0s)$, where r is the effective range. We find in this case

$$p(s) + 2i\delta^0(s) - p(0) = -2\ln\left[\frac{1 + a_0\alpha\sqrt{-s}}{1 + a_0\beta\sqrt{-s}}\right],$$
$$2\alpha = 2\beta + 2 = (1 + 2r/a_0)^{1/2} + 1,$$
$$\Gamma = -2\sqrt{s_0}/(1 - \sqrt{s_0}R),$$
$$R = 2a_0\beta/\alpha(1 + a_0\beta\sqrt{s_0}), \tag{4.20}$$
$$a = a_0 - (1 + a_0\alpha\sqrt{s_0})/\sqrt{s_0}(1 + \beta^2 a_0\sqrt{s_0}/\alpha).$$

These results again have forms very close to the rigorous ones and are nearly independent of a_0 for reasonable values of r. Thus if $f_0^{(0)}$ is known only crudely it is possible to calculate with some confidence the residue Γ and the scattering length a, if s_0 is known.

The conservative treatment given above has one fault (at least!). We have no

way of computing the binding energy s_0. The daring procedure alluded to earlier overcomes this difficulty. Consider the case of weak coupling in which the strength parameter λ is very much less than a critical value λ_c which produces a bound S-state of zero energy. If $\lambda < \lambda_c$ then there are no poles in the scattering amplitude for negative energies. The natural choice for $P(s)$ in (4.11) and (4.13) is then unity, which leads to

$$K_+(s) = -\frac{\exp\left[-p(s) - 2i\delta^0(s)\right]}{\pi} \int_0^\infty ds' \frac{\sqrt{s'}\, e^{p(s')}}{s' - s - i\epsilon}. \tag{4.21}$$

If this form for K is infinite, then $L = 1/K$ is zero and the full amplitude f equals $f^{(0)}$. For a somewhat larger value of λ but still less than λ_c K may be finite, and this might correspond to a failure of the Born series in the case where there are no bound states. Now imagine that $p(s)$ which has been evaluated for $\lambda < \lambda_c$ is continued analytically in λ to values greater than λ_c. A bound state will appear at an energy $-s_0$ such that $p(-s_0) = +\infty$. At this point K has developed a zero and thus L now has a pole corresponding to the bound state.

The requirement that $p(-s_0) = \infty$ may be expressed in a form which ties in quite closely with a method of solving for $f_0^{(0)}$ which will be developed in the next section. Let us imagine that $f_0^{(0)}(s)$ can be written in the form $N(s)/D(s)$, where D is analytic everywhere in the complex s-plane cut along the positive real axis and $N(s)$ is real for positive real s. Then

$$\delta^0(s) = -\frac{1}{2i} \ln \frac{D(s)}{D^*(s)} \tag{4.22}$$

and the integral over δ^0 which appears in p may be carried out assuming $D \to D(\infty) =$ constant as $s \to \infty$. We find

$$p(-s) = 2 \ln \frac{D(\infty)}{D(-s)}. \tag{4.23}$$

Thus if D develops a zero at $s = -s_0$ as λ increases then $p(-s_0) = +\infty$ and $f_0^{(0)}(s)$ has a simple pole and hence a bound state at $s = -s_0$ has appeared. If the procedure outlined here is correct, we see that the entire scattering and bound-state problem has been completely solved in terms of the coupling parameter λ, which is, of course, our goal.

To get some slight insight into this method let us return to the example of the effective range phase shift (see 4.20). We imagine that an analytic continuation in λ is performed. If the potential is sufficiently attractive then the originally positive scattering length a_0 will become negative and $p(s)$ becomes infinite at a negative energy given by

$$\sqrt{-s} = -\frac{1}{\alpha a_0}, \tag{4.24}$$

which is a reasonable value for such a phase shift: the binding energy is zero

for $a_0 = \infty$, as it should be and as we move away from this condition (by increasing the potential strength) the binding energy increases.

Even though we find these conjectures very attractive we cannot back them up in any firm away. If for example one were forced to make a subtraction in the dispersion relation for Im $f(s,t)$ at a finite value of t, say $t = 0$, nothing like the above procedure would work. In that case our representation becomes

$$f(s,t) = f_B(t) - t\int_0^\infty \frac{ds'}{\pi}\int_0^\infty \frac{dt'}{\pi}\frac{\rho(s',t')}{t'(t'+t)(s'-s-i\epsilon)} + \frac{\Gamma}{s+s_0} + \int_0^\infty \frac{ds'}{\pi}\frac{g_1(s')}{s'-s-i\epsilon}, \tag{4.25}$$

where g_1 is a new weight function to be determined by unitarity. The quantity analogous to $f_0^{(0)}$ in this case is no longer unitary by itself in spite of the fact that the iterative construction of ρ is unaffected by the subtraction, as is easily verified. The mapping problem for g_1 to which one is led does not seem to be soluble in any direct manner. Rather than discuss the question of finding g_1 in this form we turn to a consideration of dispersion relations for partial wave amplitudes which is closely related to the solution of this problem.

V. DISPERSION RELATIONS FOR PARTIAL WAVE AMPLITUDES

There are many reasons for discussing the analytic behavior of the partial wave amplitudes. One very important one is that the unitarity condition takes on a particularly simple form; another is that we know the asymptotic form of $e^{i\delta}\sin\delta/\sqrt{s}$ as $s \to \infty$, namely that it goes to zero. For this reason we may hope that the subtraction question is somewhat less burdensome here than it is for the full amplitude.

The analytic properties of the partial wave amplitudes do not depend on the question of subtractions in the full amplitude. In order to make this clear let us work with the once subtracted form (4.25) where we assume there is only an s-wave bound state. We write the usual expansion

$$f(s,t) = \sum_{l=0}^{\infty}(2l+1)f_l(s)P_l\left(1-\frac{t}{2s}\right), \tag{5.1}$$

where in the argument of the Legendre polynomial P_l we have written $\cos\vartheta = 1 - t/2s$ and $f_l = e^{i\delta_l}\sin\delta_l/\sqrt{s}$.

In attempting to project from (4.25) the partial wave amplitudes we encounter in f_B and in the double integral term the integral J_l, where

$$J_l = \frac{1}{2}\int_{-1}^{+1} dz\,\frac{P_l(z)}{t'+2s(1-z)}, = \frac{1}{2s}Q_l\left(1+\frac{t'}{2s}\right). \tag{5.2}$$

The Q_l's are Legendre functions of the second kind and the only properties of them we need can be read directly from (5.2). We find then from (4.25) the following set of equations:

$$\begin{aligned} f_0(s) &= -\int_m^\infty d\mu\, \sigma(\mu)\, \frac{1}{2s}\, Q_0\left(1+\frac{\mu^2}{2s}\right) + \frac{\Gamma}{s+s_0} + \int_0^\infty \frac{ds'}{\pi}\, \frac{g_1(s')}{s'-s-i\epsilon} \\ &\quad + \int_0^\infty \frac{ds'}{\pi} \int_{4m^2}^\infty \frac{dt'}{\pi}\, \frac{\rho(s',t')}{s'-s-i\epsilon} \times \frac{1}{2s}\left[Q_0(1+t'/2s) - \frac{2s}{t'}\right], \\ f_l(s) &= -\int_m^\infty d\mu\, \sigma(\mu)\, \frac{1}{2s}\, Q_l\left(1+\frac{\mu^2}{2s}\right) + \int_0^\infty \frac{ds'}{\pi} \int_{4m^2}^\infty \frac{dt'}{\pi} \\ &\quad \cdot \frac{\rho(s',t')}{s'-s-i\epsilon}\, \frac{1}{2s}\, Q_l(1+t'/2s), \end{aligned} \tag{5.3}$$

$$l > 0. \tag{5.4}$$

From these equations we may read off the analytic properties of f_0 and f_l regarding them as functions of the complex variable s. We see that in each case there is a cut along the positive real s axis from the denominator $s' - s$. There is also a cut in both along the negative s axis coming from the Q's. In fact from (5.2) we find that

$$\begin{aligned} &\frac{1}{2i}\left\{Q_l\left(1+\frac{t'}{2(s+i\epsilon)}\right) - Q_l\left(1+\frac{t'}{2(s-i\epsilon)}\right)\right\} \\ &\qquad = \frac{\pi}{2} P_l(1+t'/2s)\theta(-s-t'/4). \end{aligned} \tag{5.5}$$

Thus from the first Born term we see the cut runs from $-m^2/4$ to $-\infty$, and from the double integral, from $-m^2$ to $-\infty$.

It is easy to show that with the assumption we have made about the representation in (4.25), namely only one subtraction at $t = 0$, both f_0 and $f_l \to 0$ as $|s| \to \infty$ anywhere in the plane. In general when we have $(n+1)$ subtractions in t as in (2.26), one can show that at most n subtractions are required for each $f_l(s)$, i.e., $f_l(s)/s^n \to 0$ as $|s| \to \infty$.

We may now write the simple Cauchy formula for f_0 and f_l which embodies the above analyticity:

$$\begin{aligned} f_0(s) &= f_{0B}(s) + \frac{\Gamma}{s+s_0} + \int_0^\infty \frac{ds'}{\pi}\, \frac{\operatorname{Im} f_0(s')}{s'-s-i\epsilon} + \int_{-\infty}^{-m^2} \frac{ds'}{\pi}\, \frac{\operatorname{Im}(f'_0 - f'_{0B})}{s'-s}, \\ f_l(s) &= f_{lB}(s) + s^l \int_0^\infty \frac{ds'}{\pi}\, \frac{\operatorname{Im} f_l(s')}{s'^l(s'-s-i\epsilon)} + s^l \int_{-\infty}^{-m^2} \frac{ds'}{\pi}\, \frac{\operatorname{Im}(f'_l - f'_{lB})}{s'^l(s'-s)}, \end{aligned} \tag{5.6}$$

where the Im $f_l(s)$ for $-\infty < s < -m^2$ may be represented in terms of the weight function $\rho(s,t)$:

$$\text{Im}\,(f_l - f_{lB}) = \int_0^\infty \frac{ds'}{\pi} \int_{4m^2}^\infty \frac{dt'}{\pi} \cdot \frac{\pi}{4s} \frac{\rho(s',t')}{s' - s} P_l(1 + t'/2s)\theta[-s - t'/4], \quad -\infty < s < -m^2. \tag{5.7}$$

This last relation holds for all l and is independent of the number of subtractions in t. We note that f_{lB}, the first term in (5.3) and (5.4) also has a negative cut. We have in writing the dispersion relation for f_l considered instead f_l/s^l. This wasn't necessary but if one at any subsequent stage wants to make an approximation in the negative cut, it is useful to insure the correct low energy behavior, which is s^l. A precisely analogous discussion in the field theoretic case has been given by MacDowell (*9*).

We note that for all $l > 0$, f_l is determined by quadratures assuming $\rho(s', t')$ is known. The latter statement is legitimate in principle because we have given a construction for ρ. For f_0, however, we evidently have a nonlinear integral equation to solve. It is nonlinear because from unitarity we know Im $f_0(s) = \sqrt{s}\,|f_0(s)|^2$. One method to handle such an equation has been given by Noyes and Wong (*10*). We shall review this procedure because it ties in with our discussion in Section IV and because we can give a rigorous justification of it in this case. We write

$$f_0(s) = \frac{N(s)}{D(s)}, \tag{5.8}$$

where $N(s)$ is to be real for $s > 0$ but may have a cut along the negative s-axis; $D(s)$ is real for negative s, has a cut along the positive s axis and further $D(s) \to 1$ as $|s| \to \infty$. We may prove easily that $N(s) \to 0$ as $|s| \to \infty$ from our representation and the properties of ρ which we must have assumed, in (4.25). Except for the above-mentioned singularities $N(s)$, $D(s)$ are analytic. We may then write

$$D(s) = 1 + \int_0^\infty \frac{ds'}{\pi} \frac{\text{Im}\, D(s')}{s' - s - i\epsilon}, \qquad N(s) = \int_{-\infty}^{-m^2/4} \frac{ds'}{\pi} \frac{\text{Im}\, N(s')}{s' - s - i\epsilon}. \tag{5.9}$$

Next we remark that Im $D(s) = N(s)$ Im $(1/f_0(s))$ for $s > 0$, since $N(s)$ is real in this region. We also know that Im $(1/f_0(s)) = -\sqrt{s}$ from unitarity. To obtain Im $N(s)$, $s < -m^2/4$, we use the reality of D in this region together with (5.7) with $l = 0$:

$$\begin{aligned}\operatorname{Im} N(s) &= D(s)\frac{\pi}{4s}\left[-\int_m^\infty d\mu\,\sigma(\mu)\theta\left(-s-\frac{\mu^2}{4}\right)\right.\\ &\left.+\int_0^\infty \frac{ds'}{\pi}\int_{4m^2}^\infty \frac{dt'}{\pi}\frac{\rho(s',t')}{s'-s}\,\theta\left(-s-\frac{t'}{4}\right)\right] \qquad (5.10)\\ &\equiv \pi D(s)\alpha(s), \qquad s<-\frac{m^2}{4}.\end{aligned}$$

We now have

$$\begin{aligned}D(s) &= 1-\frac{1}{\pi}\int_0^\infty ds'\,\frac{\sqrt{s'}\,N(s')}{s'-s-i\epsilon},\\ N(s) &= \int_{-\infty}^{-m^2/4} ds'\,\frac{D(s')\alpha(s')}{(s'-s)},\end{aligned} \qquad (5.11)$$

and eliminating $N(s)$ and interchanging the orders of integration we finally achieve

$$D(-s) = 1+\int_{m^2/4}^\infty ds'\,\frac{\alpha(-s')D(-s')}{\sqrt{s'}+\sqrt{s}}. \qquad (5.12)$$

This may be transformed to a nonsingular integral equation of the Fredholm type by the substitution $s' = 1/\chi^2$. The bound states, if any, correspond to the zeros of D for negative s. The function α is given in terms of ρ in (5.10).

It is perhaps worthwhile to point out what would have happened if it had been necessary in our original representation (4.25) to make two subtractions. [We regard this possibility as intuitively unlikely if there is only an s wave bound state although if the p-wave Born series were to diverge, as well as the s-wave, it could happen.] All of the phase shifts for $l > 1$ would be computed by quadratures, but now an integral equation would have to be solved for both s and p waves. $N_0(s)$ would have to have a subtracted dispersion relation and a constant $N_0(0)$ would appear in it. The corresponding quantity $N_1(0)$ is zero for the p-wave equations. The solution to the problem would contain $N_0(0)$ which would presumably be determined by the demand that $N_0(s) \to 0$ as $s \to +\infty$ along the real axis, which is the requirement that the amplitude $f_0(s)$ should approach zero for large real positive s.

We now show that, in our case of potential scattering, $f_0(s)$ indeed has the form $N(s)/D(s)$ with N and D having the required properties. To do this we introduce the Jost (*11*) function $g(k)$ in terms of which the s matrix takes the form

$$s(k) = e^{2i\delta_0} = \frac{g(k)}{g(+k)^*} = \frac{g(k)}{g(-k)}. \qquad (5.13)$$

Jost proves that $g(k)$ is analytic in the *lower* k plane. From our discussion it

follows that the singularities and branch cuts of this function are restricted to be on the positive imaginary k-axis for a suitably restricted class of potentials.[4] Now the scattering amplitude $f(k)$ is

$$f(k) = \frac{\dfrac{g(k) - g(-k)}{2ik}}{g(-k)} \equiv \frac{N(k^2)}{g(-k)} \equiv \frac{N(k^2)}{D(k)}. \tag{5.14}$$

Evidently N depends on k^2 since it is an even function of k. Also N can have singularities and branch cuts, coming from $g(k)$, only for negative k^2. The corresponding singularities in $g(-k)$ are on the second sheet of the Riemann surface in the k^2 plane and hence are irrelevant for our considerations. $D(k)$ will have a cut on the positive k^2 axis from zero to infinity because it depends on $k = \sqrt{k^2}$. It has no singularities for negative k^2 because $g(-k)$ is well behaved for k in the upper half plane. Hence the representation we assumed for f is justified. Jost also shows that $g(\infty) = 1$ for Im $k \leqq 0$ so we have $D(k^2) \to 1$ as $k^2 \to \infty$. Finally he proves that if $g(\infty) = 1$ the bound states correspond to the zeros of $g(-k)$ with k in the upper half k plane.

We will show how the D equation can be explicitly solved for the case of the exponential potential, where the cut on the negative energy axis has degenerated into a sequence of poles. These poles are closely related to the so-called redundant zeroes of the s-matrix which have been frequently discussed in the literature.

The residues of the poles on the negative energy axis for the s-wave scattering amplitude may be computed from the known solution to this problem. We have

$$\text{Im } f_0(-s') = 2\pi \sum_{r=1}^{\infty} \frac{(-1)^{r+1} g^r}{r!\,(r-1)!}\, \delta(r^2 - 4s'), \tag{5.15}$$

where g is the strength of the potential which we take to have unit range. Inserting this into our N equation, (5.11), we have

$$N(s) = -2 \sum_{r=1}^{\infty} \frac{(-1)^{r+1} g^r}{r!\,(r-1)!}\, \frac{D(-r^2/4)}{4s + r^2} \tag{5.16}$$

and then find

$$D(s) = 1 + \frac{2}{\pi} \sum_{r=1}^{\infty} \frac{(-1)^{r+1} g^r}{r!\,(r-1)!}\, D\left(-\frac{r^2}{4}\right) \int_0^{\infty} \frac{ds' \sqrt{s'}}{(s' - s - i\epsilon)(4s' + r^2)}. \tag{5.17}$$

Evaluating the integral, we have

$$D(-s) = 1 + \sum_{r=1}^{\infty} \frac{(-1)^{r+1} g^r D(-r^2/4)}{r!\,(r-1)!\,(r + 2\sqrt{s})}. \tag{5.18}$$

[4] The analytic properties of the partial wave amplitudes have also been recently studied by Martin (*12*).

There is a difference equation which can be solved by standard methods. The answer, which can be readily checked, is

$$\begin{aligned} D(-s) &= 1 + \sum_{r=1}^{\infty} \frac{g^r}{r!} \frac{1}{(1+2\sqrt{s})(2+2\sqrt{s}) \cdots (r+2\sqrt{s})} \\ &= 1 + \sum_{r=1}^{\infty} \frac{(-1)^{r+1} g^r}{(r-1)!\,(r+2\sqrt{s})} \sum_{n=0}^{\infty} \frac{g^n}{n!\,(n+r)!}. \end{aligned} \tag{5.19}$$

The result is the correct one, and shows that in this case the N/D method works.

It is also interesting to note that the vanishing of D also leads to the correct eigenvalue condition. Since $D(\infty) = 1$ and also

$$D(-s_0) = \frac{\Gamma(1+2\sqrt{s_0})}{(-g)^{\sqrt{s_0}}} J_{2\sqrt{s_0}}(2\sqrt{-g}), \tag{5.20}$$

the vanishing of D is equivalent to

$$J_{2\sqrt{s_0}}(2\sqrt{-g}) = 0, \tag{5.21}$$

which is the well-known eigenvalue statement for the Schrödinger equation. (g is negative for an attractive potential.) This example also serves to clarify the meaning of the so-called redundant poles (*13*). They are simply the remnant of the negative energy cut in the partial wave amplitude.

Noyes and Wong (*10*) have explored the consequences of taking into account only the first Born approximation contribution to the negative cut for a pure Yukawa potential. The argument in favor of this approximation is that if one is interested in small s, say of the order of $m^2/4$ or less, the cut coming from the double integral term which starts at $-m^2$ should be of less importance than that of the Born cut which starts at $-m^2/4$. In order to test this idea we have considered the same approximation for the exponential potential for which both the approximate and exact problems are analytically soluble.

The equation for $f_0(s)$ taking into account only the Born term is (assuming no bound state)

$$f_0(s) = \frac{-2g}{1+4s} + \int_0^{\infty} ds' \frac{\sqrt{s'}\,|f_0(s')|^2}{s'-s-i\epsilon}. \tag{5.22}$$

This is a standard equation whose solution is

$$f_0 = \frac{-2g/(1+4s)}{1 + \dfrac{2g(1+4s)}{\pi} \displaystyle\int_0^{\infty} ds' \frac{\sqrt{s'}}{(1+4s')^2(s'-s-i\epsilon)}}. \tag{5.23}$$

The integrals are readily carried out and we find

$$\sqrt{s} \cot \delta_0 = -(1+4s)\left(1+\frac{g}{2}\right)\frac{1}{2g} + 2s. \tag{5.24}$$

The solution of the Schrödinger equation for the exponential potential leads to

$$s(k) = \frac{J_{2ik}(2\sqrt{-g})}{J_{-2ik}(2\sqrt{-g})} \frac{\Gamma(1 + 2ik)}{\Gamma(1 - 2ik)} (\sqrt{-g})^{-4ik}. \tag{5.25}$$

For $-g \ll 1$, the scattering length is given by

$$\frac{1}{a} = -\left(1 + \frac{5g}{8}\right)\frac{1}{2g}. \tag{5.26}$$

Thus we see that in general the validity of neglecting the negative cut must depend on the strength of the interaction and that the suggestive arguments about the unimportance of distant singularities are not necessarily quantitatively reliable.

VI. SUMMARY AND CONCLUSION

We have shown that the Mandelstam representation can be derived from the principles of nonrelativistic quantum mechanics for a wide class of potentials, namely, those which can be constructed by superimposing Yukawa potentials of different ranges. It is far from clear that our procedure sheds any light on the corresponding derivation problem in field theory; in fact it is our feeling that it does not. Of course, one does not have the Fredholm theory available in the case of interacting fields, but it is possible to give another derivation utilizing the fact that every successive Born term has a cut which moves farther out in the cosine ϑ plane; the exact remainder may then be shown to be analytic in a Lehmann ellipse of arbitrarily large size. This technique [which has subsequently been discovered by Klein (*14*) independently] looks generalizable to the field theoretic case but certainly does not work in any obvious fashion if at all.

There are a number of other places where the two problems are much closer and we feel that considerable pedagogic insight may be acquired. For example, the one-dimensional dispersion relation with exchange forces is exactly what one obtains for the two-nucleon problem when it is treated by field theoretic methods. The discussion of the iterative construction of ρ, while simpler in our problem, is also met in field theory. The analytic properties of the partial wave amplitudes are treated in a completely parallel fashion, and the calculation of binding energies would be carried out in both cases in terms of integral equations for partial waves.

There are several questions we have been unable to answer to which we would like to call attention. First, what is the precise relation between our iterative procedure and the Born series? How many subtractions are needed in any given case? A promising attack on this problem has been made by Regge[1] who has developed a method for studying the behavior of the scattering amplitude for large momentum transfer within the framework of the Schrödinger theory. Regge's work, we feel, is of great importance since it appears to be the only one thus far developed which directly correlates the strength of the interaction to the momen-

tum transfer behavior. Finally, it is evident that the equation for the partial wave amplitudes are coupled in that the same weight function ρ, appears in them all. Can we explicitly exhibit this coupling? There are probably many other questions which we couldn't answer if we could think of them.

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APPENDIX I

We now complete the proof of Section II and show that not only each term of the Fredholm series for $f'(k, \cos\theta)$ is analytic in the cut $\cos\theta$ plane, but that also the whole f' is analytic in that region.

We start with (2.13),

$$f'(k, \cos\theta) = \int_{y_0}^{\infty} dy \int_0^{2\pi} d\chi \frac{\bar{w}(k,y,\chi)}{y - \cos(\theta - \chi)}; \tag{A.1}$$

we remark that the Fourier transform of the full Green's function appearing in (2.7) also has a Fredholm expansion,

$$G(\mathbf{q}_1, \mathbf{q}_2; k) = \frac{1}{D(k)} \sum_n g_n(\mathbf{q}_1, \mathbf{q}_2; k). \tag{A.2}$$

The only remark we need to make here about the above series is that it converges uniformly for all physical $\mathbf{q}_1$, $\mathbf{q}_2$, and k. This follows immediately from the results of Ref. *6*, since the series above is, except for some factors, essentially the Fourier transform of the series representing the Fredholm resolvent. If we now substitute (A.2) in (2.7) and perform the change to new variables indicated in (2.7)–(2.13), we obtain a series expansion for the weight function $\bar{w}$,

$$\bar{w}(k,y,\chi) = \sum_{n=0}^{\infty} \bar{w}_n(k,y,\chi). \tag{A.3}$$

It follows easily from the uniform convergence of (A.2) and the nature of the steps (2.7)–(2.13) that the series in (A.3) is uniformly convergent for

$$0 \leqq \chi \leqq 2\pi, \qquad k > 0, \quad \text{and} \quad y \geqq y_0.$$

Consider the sequence of functions, f_m', defined by

$$f_m'(k, \cos\theta) = \int_{y_0}^{\infty} dy \int_0^{2\pi} d\chi \sum_{n=1}^{m} \frac{\bar{w}_n(k,y,\chi)}{y - \cos(\theta - \chi)}. \tag{A.4}$$

Each of these functions is analytic in the cut $\cos\theta$ plane. For as long as we have a finite sum we can easily interchange the orders of summation and integration in (A.4). From Section II, Part C, each of the resulting terms could be represented as in (2.23) and hence is analytic in the cut plane. We have thus to prove the uniform convergence of the sequence f'_m in the finite cut plane. To do that we look at the function $I_N{}^M(k, \cos\theta)$ given by

$$I_N{}^M(k, \cos\theta) = \int_{y_0}^{\infty} dy \int_0^{2\pi} d\chi \frac{R_N{}^M(k,y,\chi)}{y - \cos(\theta - \chi)}, \tag{A.5}$$

with

$$R_N{}^M(k,y,\chi) = \sum_{n=N}^{M} \bar{w}_n(k,y,\chi). \tag{A.6}$$

It will be sufficient to show that for any ϵ, there exists an N_0, such that

$$| I_N{}^M(k, \cos\theta) | \leqq \epsilon, \qquad \text{for } N, M \geqq N_0,$$

and any $\cos\theta$ in the plane with $|\cos\theta| \leqq Z_0$, excluding the points on the cut. This is easy to do since, using the uniform convergence of (A.3), one can always find an N_0 such that

$$| R_N{}^M(k,y,\chi) | \leqq \delta; \qquad N, M > N_0.$$

For any finite Z_0 and $|\cos\theta| \leqq Z_0$, we can always choose δ small enough to make $| I_N{}^M(k, \cos\theta) | \leqq \epsilon$. We note that $I_N M(k, \cos\theta)$, being composed of a finite number of Fredholm terms is analytic in the cut $\cos\theta$ plane.

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REFERENCES

1. H. LEHMANN, *Nuovo cimento* **10**, 579 (1958).
2. N. BOGOLIUBOV, B. MEDVEDEV, AND M. POLIVANOV, lecture notes, translated at the Institute for Advanced Study, Princeton, 1957 (unpublished);
   H. J. BREMERMANN, R. OEHME, AND J. G. TAYLOR, *Phys. Rev.* **109**, 2178 (1958).
3. S. MANDELSTAM, *Phys. Rev.* **112**, 1344 (1958).
4. N. N. KHURI, *Phys. Rev.* **107**, 1148 (1957).
5. D. V. WIDDER, "The Laplace Transform," p. 315 ff. Princeton University Press, 1956.
6. R. JOST AND A. PAIS, *Phys. Rev.* **82**, 840 (1951).
7. J. HAMILTON, *Phys. Rev.* **114**, 1170 (1959).
8. M. L. GOLDBERGER, I. NAMBU, AND R. OEHME, *Ann. Phys.* **2**, 226 (1957).
9. S. MACDOWELL, *Phys. Rev.* (in press).
10. P. NOYES, AND D. Y. WONG, *Phys. Rev. Letters* **3**, 191 (1959).
11. R. JOST, *Helv. Phys. Acta* **22**, 256 (1947). Also see V. BARGMANN, *Phys. Rev.* **75**, 301 (1949).
12. A. MARTIN, CERN preprint.
13. R. JOST AND W. KOHN, *Phys. Rev.* **87**, 167 (1952);
    S. T. MA, *Phys. Rev.* **71**, 195 (1946).
14. A. KLEIN, preprint.

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Bound States, Shadow States and Mandelstam Representation.

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Summary. — In a previous paper a technique involving complex angular momenta was used in order to prove the Mandelstam representation for potential scattering. One of the results was that the number of subtractions in the transmitted momentum depends critically on the location of the poles (shadow states) of the scattering matrix as a function of the complex orbital momentum. In this paper the study of the position of the shadow states is carried out in much greater detail. We give also related inequalities concerning bound states and resonances. The physical interpretation of the shadow states is then discussed.

1. – Introduction.

The validity of the Mandelstam representation for the potential scattering of two spinless particles has now been firmly established for a class of generalized Yukawa potentials ([1-4]). Double dispersion relations can now be proved in a variety of methods each one adding an interesting angle to the overall picture of the analytic properties of the scattering amplitude.

In spite of these advances many details are missing and in particular little is known of the connection between bound states and subtraction terms in the transmitted momentum.

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([1]) T. REGGE: *Nuovo Cimento*, **14**, 951 (1959).

([2]) R. BLANKENBECLER, M. L. GOLDBERGER, M. N. KHURI and S. B. TREIMAN: *Ann. of Phys.*, **10**, 62 (1960).

([3]) A. KLEIN: *Journ. Math. Phys.*, **1**, 41 (1960).

([4]) L. FONDA, L. A. RADICATI and T. REGGE: *Mandelstam representation for the non-relativistic many channel problem* (in proof).

BLANKENBECLER *et al.* [2] have shown how unitarity and Mandelstam representation can replace the Schrödinger equation in the construction of the scattering matrix under some simplifying hypothesis on bound states and subtraction terms. Their analysis would attain full generality and beauty if these restrictions could be removed and their statements rigorously backed.

In this respect part of the gap is filled in a previous paper [1] of the author where it is shown that, under some restrictive hypothesis on the potential, the number of needed subtractions is finite.

In the present work explicit bounds are placed on L and some restrictions on the potential are removed. Similar bounds are also derived for bound states. Finally a simple physical interpretation is given for the shadow states.

2. – Definitions and previous results.

In [1] complex angular momenta were employed in the proof of Mandelstam representation. We summarize here briefly the most important definitions and results needed for the proof. The starting point is the partial wave equation

$$D(E, \beta)\varphi = 0 \,, \tag{2.1}$$

where

$$D(E, \beta) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + E + \left(\beta + \frac{1}{4}\right) x^{-2} - V(x) \,,$$

and

$$\beta = -(l + \tfrac{1}{2})^2 \,.$$

Eq. (2.1) wili be considered for generally complex values of β. $l = \sqrt{-\beta} - \frac{1}{2}$ will be defined as that branch of the function which is one valued in the β plane with the cut $0 \leqslant \beta < \infty$ and such that $\mathrm{Re}\,(l+\frac{1}{2}) > 0$. The « regular » solution φ of eq. (2.1) is defined by the following boundary conditions:

$$\begin{cases} \varphi(E, \beta) \sim \cos\left[x\sqrt{E} - \dfrac{\pi\sqrt{-\beta}}{2} + \delta(\beta)\right], & x \to \infty \,, \\ \varphi(E, \beta) \sim C x^{l+1} \,, & x \to 0 \,, \end{cases} \tag{2.2}$$

φ defines therefore an asymptotic phase shift $\delta(\beta)$ which reduces to the physical values when l is integer. In [1] was shown that $S(\beta) = \exp\,[2i\delta(\beta)]$ has the following properties:

a) $S(\beta)$ is meromorphyic in the β plane with the exception of a discontinuity across the cut $0 \leqslant \beta < \infty$.

b) $S(\beta)S^+(\beta) = 1$ where $S^+(\beta) = (S(\beta^*))^*$. This is equivalent to $\delta(\beta) = \delta^+(\beta)$.

In particular $\delta(\beta)$ is real if β is negative.

c) If β is negative $\dfrac{\partial\delta(\beta)}{\partial\beta} > \dfrac{\pi}{2}\sqrt{-\beta}$.

d) $\operatorname{Im}\sqrt{-\beta}[\pi \operatorname{Im}\sqrt{-\beta} - 2 \operatorname{Im}\delta(\beta)] > 0$.

Condition *d*) implies that $S(\beta)$ can have poles in $\operatorname{Im}\beta < 0$ only. *c*) shows that if there is a pole in β there is a zero in β^*.

e) $\lim\limits_{|\beta|\to\infty} \exp[-i\pi\sqrt{-\beta}][S(\beta) - 1] = 0$, $\qquad \arg\beta \geqslant 0$.

If in addition to (2.1) we suppose that $V(x)$ is a generalized Yukawa potential

$$V(x) = \int_{\mu}^{\infty} \mathrm{d}m\, \sigma(m) \frac{\exp[-mx]}{x}, \tag{2.3}$$

we find that *e*) can be improved as follows:

e')
$$\begin{cases} S(\beta) - 1 = 0 \left(\exp[i\varepsilon\sqrt{-\beta} - \alpha\sqrt{-\beta}]\right), & \arg\beta \geqslant 0, \\ \cosh\alpha = 1 + \dfrac{\mu^2}{2E}, & \varepsilon > 0 \text{ and small.} \end{cases}$$

Properties *a*) ... *e'*) are then used in connection with the partial wave expansion of the scattering amplitude

$$f(z) = \frac{1}{2i\sqrt{E}} \sum_l (2l+1)\left(\exp[2i\delta_l] - 1\right) P_l(z); \qquad z = \cos\theta. \tag{2.4}$$

This expansion is known to converge in the small Lehmann ellipse $\operatorname{Im}\theta < \alpha$. According to WATSON we can transform it into the integral

$$f(z) = \frac{1}{4\sqrt{E}} \int_C \mathrm{d}\beta [S(\beta) - 1] P_{\sqrt{-\beta} - \frac{1}{2}}(-z) \frac{1}{\cos\pi\sqrt{-\beta}}. \tag{2.5}$$

The contour C loops around the poles of $(\cos\pi\sqrt{-\beta})^{-1}$ which occur on the physical values of β on the negative axis. C must not enclose any singularity of $S(\beta)$. It is possible to deform C around the positive axis and transform

(2.5) into

$$\sqrt{E}f(z) = \frac{i\pi}{2}\sum_n S_n P_{l_n}(-z)\,2l_n+1)\,\frac{1}{\sin \pi l_n} + \tag{2.6}$$

$$+\frac{1}{4}\int_0^\infty d\beta[S(\beta+i\varepsilon)-S(\beta-i\varepsilon)]P_{\sqrt{-\beta}-\frac{1}{2}}\frac{1}{\cos \pi\sqrt{-\beta}},$$

where S is the residuum of $S(\beta)$ in the pole $\beta_n = -(l_n+\frac{1}{2})^2$. The summation is carried out on all poles of $S(\beta)$. The functions $P_{l_n}(-\cos\theta)$, where l_n is not an integer, are Legendre functions. It is apparent from (2.6) and (2.4) that $f(\cos\theta)$ is analytic in the $\cos\theta$ plane with the cut $\cosh\alpha < \cos\theta < \infty$.

The contribution of the integral can be shown to vanish when $\operatorname{Im}\theta \to \infty$ (or $\cos\theta \to \infty$). The function $P_{l_n}(-\cos\theta)$ instead of the order of $\cos\theta^{\operatorname{Re}(l_n)}$. If the set is bounded above by L' the behavior of $f(\cos\theta)$ will be dominated by $\cos\theta^{L'}$. If $L' = \infty$ $f(\cos\theta)$ will have an essential singularity at $\cos\theta = \infty$. In (1) we proved that L' is finite if $|y^2 \operatorname{Im} V(iy)| < A < \infty$. In this paper we shall give an explicit estimate for L'. The number of subtractions needed in the Mandelstam representation is then the mallest integer $L > L'$. In (2) it is proved that unitarity requires $L > K$, where K is the largest angular momentum for which there is a bound state.

3. – Bound states.

A bound state is defined as a solution of the equation

$$D(E,\beta)\psi = 0 \qquad \text{where } E,\ \beta < 0;\ (l \text{ integer}) \tag{3.1}$$

which is L^2 in $0 \ldots \infty$. We shall name bound states the solutions of (3.1) with the same boundary conditions also when l is any positive number. L^2 implies the asymptotic behaviour

$$\begin{cases} \psi \sim cx^{l+1} & x \to 0\,, \\ \psi \sim \exp[-\sqrt{-E}x] & x \to \infty\,. \end{cases} \tag{3.2}$$

In order to have a bound state E and β must be related by some equation $g(E,\beta) = 0$, where g depends on V. This equation, if solved in E, yields $E = f(\beta)$ where f is a multivalued function since there may exist several bound states with the same β.

For definiteness we choose the branch with the largest binding energy.

It is clear that in general $f(\beta)$ is differentiable. Write now the equation

$$\int_0^\infty \mathrm{d}x\, \psi D(E, \beta)\psi = 0 . \tag{3.3}$$

After an integration by parts and some algebra (3.3) becomes

$$\int_0^\infty \mathrm{d}x \left[\left(\psi' - \frac{\psi}{2x}\right)^2 + |E|\psi^2 + |\beta|\frac{\psi^2}{x^2}\right] = -\int_0^\infty V(x)\psi^2\,\mathrm{d}x . \tag{3.4}$$

A necessary condition for the existence of bound states is that

$$|E| + (|\beta|/x^2) + V(x)$$

is somewhere negative. This implies that somewhere

$$|E| + (|\beta|/x^2) \leqslant V(x).$$

This latter condition is of course the same as in classical mechanics if $l + \frac{1}{2}$ is the classical angular momentum.

If V is repulsive, *i.e.* $V \geqslant 0$, there are no bound states. Suppose now that V satisfies an inequality of the kind

$$|V(x)| \leqslant \frac{M_\alpha}{x^{2\alpha}} . \tag{3.5}$$

By derivation it can be checked that

$$P(\lambda) = \frac{|E|}{\lambda^\alpha} + \lambda^{\alpha-1}\frac{|\beta|}{x^2} , \tag{3.6}$$

satisfies the inequality

$$P(\lambda) \geqslant \alpha^{-\alpha}(1-\alpha)^{\alpha-1}|E|^{1-\alpha}\left|\frac{\beta}{x}\right|^\alpha ,$$

putting $\lambda = 1$ one finds

$$|E| + \frac{|\beta|}{x^2} \geqslant \left|\frac{E}{1-\alpha}\right|^{1-\alpha}\left|\frac{\beta}{\alpha}\right|^\alpha x^{-2\alpha} .$$

Comparing this result with (3.4) we get

$$\left|\frac{\beta}{\alpha}\right|^\alpha \left|\frac{E}{1-\alpha}\right|^{1-\alpha} \leqslant M_\alpha . \tag{3.7}$$

Some consequences of this formula are

$$(3.8) \qquad |E| < M_0; \qquad l+\frac{1}{2} \leqslant K+\frac{1}{2} \leqslant \sqrt{M_1}; \qquad |E| \leqslant \frac{M_{\frac{1}{2}}^2}{4(l+\frac{1}{2})^2}.$$

The latter can be compared with the exact result for the Coulomb potential $V = -(M_{\frac{1}{2}}/x)$

$$|E| \leqslant \frac{M_{\frac{1}{2}}^2}{4(l+1)^2}.$$

Finally by integration of the identity

$$\psi D(E,\beta)\frac{\partial\psi}{\partial\beta} - \frac{\partial\psi}{\partial\beta} D(E,\beta)\psi = \frac{\psi^2}{x^2} + \frac{\mathrm{d}E}{\mathrm{d}\beta}\psi^2,$$

n the interval $0 \ldots \infty$ we find

$$3.\ 9) \qquad \frac{\mathrm{d}E}{\mathrm{d}\beta}\int_0^\infty \psi^2\,\mathrm{d}x = -\int_0^\infty \frac{\psi^2}{x^2}\,\mathrm{d}x.$$

Eq. (3.9) can be inserted in eq. (3.4) thus obtaining after some algebra

$$(3.10) \qquad P\left(\frac{\mathrm{d}\beta}{\mathrm{d}E}\right) \leqslant M_\alpha \frac{\mathrm{d}E}{\mathrm{d}\beta}.$$

If $\alpha = 0$ or $\alpha = 1$ this inequality can be integrated and the result is

$$(3.11) \qquad \begin{cases} |E(l')| < |E(l)| \dfrac{M_1 - (l'+\frac{1}{2})^2}{M_1 - (l+\frac{1}{2})^2}, & l' > l. \\ |E(l)| < M_0 - \dfrac{(l+\frac{1}{2})^2}{(l'+\frac{1}{2})^2}[M_0 - |E(l')|]. \end{cases}$$

4. – The shadow and resonance states.

We know from the results of ref. [1] that if β_0 is a pole of $S(\beta)$ there is a solution ψ of the Schrödinger equation

$$(4.1) \qquad D(E,\beta_0)\psi = 0, \qquad E > 0,\ \mathrm{Im}\,\beta_0 < 0,$$

with the boundary conditions:

$$(4.2) \qquad \begin{cases} \psi \sim c x^{l_0+1} & x \to 0, \qquad l_0 + \frac{1}{2} = \sqrt{-\beta^0}, \\ \psi \sim \exp[i\sqrt{E}x] & x \to \infty. \end{cases}$$

If $V(x)$ is of the form (2.3) we can continue it for complex values of x in the domain $\mathrm{Re}\,(x) \geqslant 0$. According to a general theorem (see ref. [5]) ψ can also be continued in the same region. Let us consider the function $\psi(iy) = \chi(y)$. χ satisfies the differential equation

$$D(E,\beta)\chi = \ddot{\chi} - E\chi + (\beta + \tfrac{1}{4})y^{-2}\chi + V(iy)\chi = 0\,, \tag{4.3}$$

where dots mean y-derivatives. From $\int \chi^*(D(E,\beta_0)\chi)\,\mathrm{d}y = 0$ by using the same techniques of Section **3** we find

$$\mathrm{Im}\,\beta \int_0^\infty |\chi|^2 \frac{\mathrm{d}y}{y^2} + \int_0^\infty \mathrm{d}y\,|\chi|^2\,\mathrm{Im}\,V(iy) = 0\,, \tag{4.4}$$

$$\int_0^\infty \left[\left| \dot{\chi} - \frac{\chi}{2y} \right|^2 + E|\chi|^2 - \mathrm{Re}\,\beta\,\frac{|\chi|^2}{y^2} - \mathrm{Re}\,(V(iy))\,|\chi|^2 \right] \mathrm{d}y = 0\,, \tag{4.4'}$$

$E - \mathrm{Re}\,\beta - \mathrm{Re}\,V$ must be negative somewhere in $0 \ldots \infty$.

In discussing these integral we must remember that for large positive y we have $\chi \sim \exp[-\sqrt{E}y]$. χ is therefore L^2.

If we assume now that $V(iy)$ satisfies the set of inequalities

$$\begin{cases} y^{2\alpha}|\mathrm{Im}\,V(iy)| < N'_\alpha\,; & y^{2\alpha}|\mathrm{Re}\,V(iy)| < N''_\alpha\,, \\ y^{2\alpha}|V(iy)| < N_\alpha\,; & N_\alpha < |N'_\alpha + iN''_\alpha|\,, \end{cases} \tag{4.5}$$

we can draw the following conclusions by using the same methods of Section **3**:

$$|\mathrm{Im}\,\beta| < N'_1; \qquad \left|\frac{E}{1-\alpha}\right|^{1-\alpha} \left|\frac{\mathrm{Re}\,\beta}{\alpha}\right|^{\alpha} \leqslant N''_\alpha \quad \text{if} \quad \mathrm{Re}\,\beta < 0\,, \tag{4.6}$$

$$\text{no restrictions} \quad \text{if} \quad \mathrm{Re}\,\beta > 0\,.$$

Condition (4.6) provides some bounds on L if N is finite. We have for instance $L \leqslant \mathrm{Re}\,\sqrt{N''_1 + iN'_1} - \tfrac{1}{2}$. The interesting case of Yukawa potential does not admit a finite N'_1 and we must therefore use a more sophysticated technique. To this purpose let us multiply (4.4) by $\cos\eta$, (4.4') by $\sin\eta$, and add, where $0 < \eta < \pi/2$. The result to be used only when $(\cos\eta\,\mathrm{Im}\,\beta + \sin\eta\,\mathrm{Re}\,\beta) \leqslant 0$ is

$$\left(\frac{E}{1-\alpha}\right)^{1-\alpha} \sin\eta^{1-\alpha} \left|\frac{\cos\eta\,\mathrm{Im}\,\beta + \sin\eta\,\mathrm{Re}\,\beta}{\alpha}\right|^{\alpha} \leqslant N_\alpha\,. \tag{4.7}$$

---

[5] E. T. Whittaker and G. N. Watson: *A Course of Modern Analysis* (Cambridge, 1952).

In the β plane cond. (4.5) is in general different from (4.6). If equality would hold in (4.5) this equation would represent a set of straight lines in the β plane. The permissible domain lies then below the real axis and above the envelope of these lines. For $\alpha = 0$ there is no envelope. For $\alpha = 1$ the envelope is the circle $|\beta| = N_1$, with $0 \leqslant \arg \sqrt{-\beta} \leqslant \pi/2$, and the line $\operatorname{Im}\beta = -N$. The corresponding conditions are $0 > \operatorname{Im}\beta > -N_1$ and $\operatorname{Re}\beta > > -\sqrt{N^2 - \operatorname{Im}\beta^2}$. If $\alpha = \frac{1}{2}$ the envelope is most simply expressed in the plane as

$$\operatorname{Re} l < \frac{N_{\frac{1}{2}}}{2\sqrt{E}} - \frac{1}{2}. \tag{4.8}$$

This condition can be easily applied to the Yukawa potential. The other values of α require a much more involved algebra and provide a bound only if $\alpha > \frac{1}{2}$.

The same procedure can be applied to the resonance states. The resonance state can be defined as the solution of (4.1) with the boundary conditions (4.2) but with l real and integer and E complex. The branch of $\sqrt{E}$ has to be choosen in order to have $\operatorname{Im} E < 0$, this condition follows from the continuity equation. We find instead of (4.6)

$$\operatorname{Im} E < N_0'; \qquad \left|\frac{\operatorname{Re} E}{1-\alpha}\right|^{1-\alpha} \left|\frac{\beta}{\alpha}\right|^{\alpha} < N_\alpha'', \qquad \text{if} \quad \operatorname{Re} E > 0, \tag{4.9}$$

otherwise no restriction.

And instead of (4.7)

$$\left|\frac{\beta \sin\eta}{\alpha}\right|^{\alpha} |\cos\eta \operatorname{Im} E + \sin\eta \operatorname{Re} E|^{1-\alpha} < N_\alpha. \tag{4.10}$$

The allowed areas for the resonances in the E plane are for $\alpha = 0$ the union of the circle $|E| < N_0$ and of the strip $\operatorname{Re} E < 0$. If $\alpha = \frac{1}{2}$ and $k^2 = E$ we have $\operatorname{Re} k < N_{\frac{1}{2}}/(2l+1)$.

5. – The physics of the shadow states.

It can be easily seen that shadow states are closely connected with resonances. Take namely the formula

$$\frac{\mathrm{d}E}{\mathrm{d}\beta} = -\frac{\int_0^\infty \psi^2 x^{-2} \mathrm{d}x}{\int_0^\infty \psi^2 \mathrm{d}x}. \tag{5.1}$$

Eq. (5.1) is a close analog of (3.9). Let we suppose now that $\mathrm{Im}\,\beta$ is very small. *A* small variation $\Delta\beta$ in β produces a small change ΔE in E. We choose $\Delta\beta = -i\,\mathrm{Im}\,\beta$ so that $\beta' = \Delta\beta + \beta$ is real. Correspondingly the energy will become complex $E + \Delta E = E + i\,\mathrm{Im}\,\beta/(R^2)$, $1/R^2$ being the average value of $1/x^2$ in the shadow state. These rough arguments show that there should be a resonance at the energy E if $\mathrm{Re}\,l$ is integer. The mean life of the resonance is given by $1/\Delta E$ or $R^2/\mathrm{Im}\,\beta$ having adopted units such that $\hbar = c = 2m = 1$. What physical meaning can we associate with $\Delta\beta$? Classically a resonance is a scattering process where the path of the incoming particle winds up several times around the center of force and has therefore the opportunity of interacting for a long period with the potential. We can suppose that the radius of the orbit can be conveniently approximated by R. If l is the orbital momentum the angular speed $\mathrm{d}\varphi/\mathrm{d}t$ is $2l/R^2$.

During the interaction time the particle travels the angle $\Delta\varphi = \Delta t \cdot \mathrm{d}\varphi/\mathrm{d}t = 1/\mathrm{Im}\,l$ before it is lost again in the scattering process. The above estimate can be also checked as follows. We write the total wave function as

$$\Psi = \frac{1}{r}\, Y_l^m(\theta, \varphi)\, \psi(r)\,. \tag{5.2}$$

Of course Ψ is not a true wave function since it is multivalued in the angles. However if we disregard these irregularities it provides a fairly accurate description of the scattering process when we are sufficiently far from the branch points of Y_l^m. Take now $m = l$. We have $Y_l^l = P_l^l(\theta)\exp[il\varphi]$. This function is multivalued in φ and $\mathrm{Im}\,l$ provides a damping factor in the orbit which will allow the particle to travel by an angle $\Delta\varphi$ of the same order of magnitude as found before. We can confidently assert that a shadow state with a very small $\mathrm{Im}\,l$ originates from a resonance in the energy. In writing up a Mandelstam representation of a given process the number of subtractions L has to be equal or larger than the angular momentum of the intermediate resonance. The above arguments lose their strength if $\mathrm{Im}\,\beta$ is no longer small because then the life-time of the resonance becomes too short.

An interesting interpretation of the shadow states, from which the name derives, follows from the work of LEVY and KELLER on the theory of diffraction (6). These authors have shown that in calculating the diffraction of a field against a smooth object it is possible to modify the ordinary geometrical optics, and obtain reliable results, by introducing new kind of rays, the diffracted rays, besides the optical ones.

The diffracted rays are produced by incident rays which are tangent to the surface of the body. A part of the incident ray, instead of continuing along

(6) B. LEVY and J. KELLER: *Diffraction on a smooth object.*

the geometrical path, follows a geodesics along the body, until it leaves eventually the body along a tangent. The amplitude of the part of the ray which undergoes this process decreases exponentially with the arc of geodesics travelled by the diffracted ray. Diffracted rays obviously penetrate into the geometrical shadow of the object. By analyzing the scattering amplitude in terms of complex angular momenta LEVY and KELLER prove that diffracted rays arise from the shadow states. In their case Im is essentially a angular penetration factor of the ray into the shadow, thereby justifying the name shadow state. Unfortunately there is no immediate generalization of Levy and Keller's theory of diffraction to the simple potential scattering. It has to be pointed out that the boundary condition for a hard core potential, corresponding to a mooth object, yield to somewhat simple analytic properties of $S(\beta)$, in fact in this case $S(\beta)$ is meromorphyc in β, there being no cut in $0 < \beta < \infty$.

In view of the considerable amount of work being done on phenomenological hard core potentials for nucleon-nucleon scattering it would be desirable to extend our results to more realistic cases.

* * *

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RIASSUNTO

In questo lavoro si estendono i risultati di una nota precedente dell'autore in cui si è fatto uso di momenti angolari complessi. In particolare vengono derivate ineguaglianze concernenti il numero di sottrazioni nelle regole di dispersione alla Mandelstam per vari tipi di potenziale tra cui il potenziale di Yukawa.

COMMENTS AND CORRECTIONS

Dr. Cutkowky has requested that the following statement regarding his paper be noted:

In the discussion contained in Sec. IIB, the possibility that for some of the $i \leq m$ *both* α_i *and* A_i may be zero was overlooked. It is easiest to examine this possibility along the lines of Sec. IIA; it is found to occur when the setting equal to zero of some of the β_i is required in order to ensure the compatibility of Eqs. (8). Although the singularities and discontinuities of $F_m(z)$ are indeed associated with graphs that have additional lines, such a singularity might have the same locus as a singularity of $F(z)$ that corresponds to a graph with fewer lines.

The discussion following Eq. (12) tacitly assumes that either $b_m \rightarrow M_m^2$ from the left or $a_m \rightarrow M_m^2$ from the right. This implies a particular sign for J, which must be remembered when the sign of F_m is calculated. In Eq. (6), the sign of J is hidden in the prescription to take the "proper" root of $q_i^2 M_i^2$. This means that Eq. (12) does not actually allow one to circumvent any hard work.

Professor Regge has requested that the following corrections to his paper be noted:

3 Line 4: For "c) shows" read "b) shows."

4 Line 14: For "mallest" read "smallest."

5 Line 6: For "$|E| + (|\beta|/x^2) + V(x)$" read "$|E| + |\beta|/x^2 + V(x)$."

Line 8: For "$|E| + (|\beta|/x^2) \leq V(x)$" read "$|E| + |\beta|/x^2 \leq -V(x)$."

5 Eq. (3.6): For "$P(\lambda) = \frac{|E|}{\lambda^{\alpha}} + \lambda^{\alpha-1}\frac{|\beta|}{x^2}$," read

"$P(\lambda) = \frac{|E|}{\lambda^{\alpha}} + \lambda^{1-\alpha}\frac{|\beta|}{x^2}$."

Line 17: For "$P(\lambda) \geq \alpha^{-\alpha}(1-\alpha)^{\alpha-1}|E|^{1-\alpha}\left|\frac{\beta}{x}\right|^{\alpha}$," read

"$P(\lambda) \geq \alpha^{-\alpha}(1-\alpha)^{\alpha-1}|E|^{1-\alpha}\left|\frac{\beta}{x^2}\right|\alpha$."

7 Line 10: For "$E - \mathrm{Re}\,\beta - \mathrm{Re}\,V$" read "$E - \frac{\mathrm{Re}\,\beta}{y^2} - \mathrm{Re}\,V$."

Line 22: For "$\cos\eta$" read "$-\cos\eta$."

8 Line 1: For "(4.5)" read "(4.7)."

Line 6: For "$\mathrm{Im}\,\beta = -N$" read "$\mathrm{Im}\,\beta = -N_1$."

Line 16: For "$\mathrm{Im}\,E < 0$" read "$\mathrm{Im}\sqrt{E} < 0$."

Eq. (4.9): For "$\mathrm{Im}\,E < N_0'$" read "$|\mathrm{Im}\,E| < N_0'$."

Eq. (4.10): For "$|\cos\eta\,\mathrm{Im}\,E + \sin\eta\,\mathrm{Re}\,E|^{1-\alpha}$" read

"$\left|\frac{\cos\eta\,\mathrm{Im}\,E - \sin\eta\,\mathrm{Re}\,E}{1-\alpha}\right|^{1-\alpha}$."

10 Line 12: For "mooth" read "smooth."

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